

# Bounded Degree Cosystolic Expanders of Every Dimension

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## Abstract

In recent years a high dimensional theory of expanders has emerged. The notion of combinatorial expansion of graphs (i.e. the Cheeger constant of a graph) has seen two generalizations to high dimensional simplicial complexes. One generalization, known as *coboundary expansion*, is due to Linial and Meshulam; the other, which we term here *cosystolic expansion*, is due to Gromov, who showed that cosystolic expanders have the topological overlapping property. No construction (either random or explicit) of *bounded degree* combinatorial expanders (according to either definition) were known until a recent work of [KKL]. The work of [KKL] provided the first bounded degree cosystolic expanders of dimension two. No bounded degree combinatorial expanders are known in higher dimensions.

In this work we present explicit *bounded degree* cosystolic expanders of *every dimension*. This solves affirmatively an open question raised by Gromov, who asked whether there exist bounded degree complexes with the topological overlapping property in every dimension.

Moreover, we provide a local to global criterion on a complex that implies cosystolic expansion: Namely, if the 1-skeleton graph underlying a  $d$ -dimensional complex  $X$  is a good expander graph and all its links are both coboundary expanders and good expander graphs, then the  $(d - 1)$ -dimensional skeleton of the complex is a cosystolic expander.

## 1 Introduction

Expander graphs have been central objects of study both in computer science and pure mathematics, in the past few decades, with numerous applications (see [HLW], [L1]). In recent years, a new theory of high dimensional expanders has emerged, pioneered by the works of Linial-Meshulam [LM], and Gromov [G] (for a recent survey see [L2]). Linial-Meshulam and Gromov suggested two generalizations of the notion of combinatorial expansion of a graph (i.e. the Cheeger constant of a graph) to higher dimensions. One is known as coboundary expansion (à la Linial and Meshulam) and the other is termed here cosystolic expansion (à la Gromov).

A graph (resp. a complex) is considered as *bounded degree*, if the number of edges (resp. faces) incident to every vertex is independent of the total number of vertices in the graph (resp. in the complex). Much of the study of expander graphs has focused on constructions of families of *bounded degree* graphs with strong expansion properties. However in the high dimensional case, no constructions (either random or explicit) of *bounded degree* combinatorial expanders (according to either definition) were known until a recent work of [KKL], which provided the first bounded degree cosystolic expanders of dimension two. Assuming the Serre conjecture, the work of [KKL] also implies the first bounded degree coboundary expanders of dimension two.

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In this work we extend the ideas of [KKL], and derive for the first time, bounded degree cosystolic expanders of every dimension. As a consequence, we provide an affirmative answer to an open question raised by Gromov [G], who asked about the existence of bounded degree complexes with the topological overlapping property (see below). In fact, we provide a general local to global criterion, on a complex, that implies cosystolic expansion. We use this criterion to construct explicit families of bounded degree cosystolic expanders of every dimension.

## 1.1 Expander Graphs

Following we review some basic properties of expander graphs (for more on expander graphs see the excellent surveys [HLW] and [L1]). Later we will discuss analogues of these properties in the high dimensional case. Throughout this section,  $G = (V, E)$  is a finite  $d$ -regular graph.

### 1.1.1 Strong Connectivity

Expander graphs were defined explicitly by Pinsker [Pin] in 1973 (who coined the name), as bounded degree graphs which are strongly connected. For  $S \subset V$ , define its boundary,  $\delta(S) = E(S, \bar{S})$ , to be the subset of edges with one endpoint inside  $S$  and the other outside  $S$ . A graph  $G$  is connected if and only if  $|\delta(S)| > 0$  for every  $S \subset V$  which is not trivial, i.e.  $S \neq \emptyset$  or  $V$ . The strong connectivity property of a graph is measured by its, so called, *Cheeger Constant*,

$$h(G) = \min_{\emptyset \neq S \subsetneq V} \frac{|E(S, \bar{S})|}{\min(|S|, |\bar{S}|)}.$$

A graph  $G$  is called an  $\epsilon$ -combinatorial expander if  $h(G) \geq \epsilon$ .

### 1.1.2 Spectrum and Pseudorandomness

Expander graphs are pseudorandom, i.e. they behave similarly to random graphs. The pseudorandomness of a graph is measured by the spectrum of its adjacency matrix. Let  $A_G$  be the adjacency matrix of the graph  $G$ , and define the second largest eigenvalue in absolute value,  $\lambda(G)$ , of a  $d$ -regular graph  $G$ , by:

$$\lambda(G) = \max\{|\lambda| \mid \lambda \text{ an eigenvalue of } A_G \text{ and } \lambda \neq \pm d\}.$$

A graph  $G$  is said to be an  $\epsilon$ -spectral expander if  $\lambda(G) \leq d - \epsilon$ . The Expander Mixing Lemma implies that the smaller  $\lambda(G)$  is, the better the pseudorandomness behavior of the graph is. Namely,

**Lemma 1.1** (Expander Mixing Lemma). *Let  $G = (V, E)$  be a  $d$ -regular graph, which is not bipartite. For any  $S, T \subseteq V$ , then*

$$\left| |E(S, T)| - \frac{d \cdot |S| \cdot |T|}{|V|} \right| \leq \lambda(G) \cdot \sqrt{|S||T|},$$

where  $|E(S, T)|$  is the number of edges between  $S$  and  $T$ .

### 1.1.3 Spectral and Combinatorial Expansion

A very useful fact in expander graph theory, is that spectral expansion (namely, being pseudorandom, i.e having a small second largest eigenvalue in absolute value), implies combinatorial

expansion (namely, being strongly connected, i.e. having a large Cheeger constant). This relation is given by the following Cheeger inequality: <sup>1</sup>  $h(G) \geq \frac{d - \lambda(G)}{2}$ .

The important relation between spectral and combinatorial expansion that holds for graphs stops to hold once moving to higher dimensional expanders. This implies some of the mystery (and difficulty) in the study of high dimensional analogues of expanders. In particular, the non existence of such high dimensional relation between combinatorial and spectral expansion, sheds some light on the difficulty of obtaining bounded degree high dimensional combinatorial expanders, which are the focus of this work.

#### 1.1.4 Embedding Complexity

Recently, it was observed (see [L1] and the references therein) that Barzdin and Kolmogorov [BK] considered a property of graphs which is equivalent to expanders, already in 1967. The motivation of [BK] was to study the question of embedding graphs into Euclidean spaces. Essentially, what Barzdin and Kolmogorov showed was that expander graphs do not embed easily into  $\mathbb{R}^3$ , and that random graphs are with high probability expander graphs. This notion of expansion is going to be strongly related to the notion of high dimensional expansion à la Gromov (i.e. the notion of cosystolic expansion) that we study in this work.

### 1.2 High Dimensional Expanders - Some Motivations

In recent years a high dimensional study of expansion has emerged, where the object of study has switched from graphs (which are one dimensional simplicial complexes) to higher dimensional simplicial complexes. There are two commonly studied generalizations of the notion of combinatorial expansion to higher dimensions; one is known as coboundary expansion and the other is termed here cosystolic expansion. Before introducing the exact (slightly technical) definitions of high dimensional expanders, let us begin with some motivations for studying them.

#### 1.2.1 Cohomological Connectivity

For random graphs (in the Erdos-Renyi model) the connectivity property has a threshold phenomena. Namely, below a certain probability (the threshold), a  $G(n, p)$  graph is almost surely disconnected, while above the threshold, the random graph is almost surely connected, and in fact, it is strongly connected, i.e. an expander.

In [LM] and [MW], the authors developed a model of random complexes which generalizes the Erdos-Renyi model for graphs, to higher dimensional simplicial complexes. An analogue threshold phenomena for connectivity for simplicial complexes, has been proven in [LM] and [MW], where a  $d$ -dimensional complex is said to be connected if its  $(d - 1)$ -dimensional  $\mathbb{F}_2$ -cohomology vanishes. Namely, below a certain probability (the threshold), a random complex in the Linial-Meshulam model is almost surely disconnected, while above that threshold, the complex is almost surely connected, and in fact, it satisfies a stronger connectivity property, which is the coboundary expansion (see below). For  $d = 1$ , i.e. for graphs, vanishing of the 0-dimensional cohomology is equivalent to being connected, and coboundary expansion is equivalent to graph expansion.

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<sup>1</sup>We note that for a slightly weaker notion of spectral expansion, namely the second largest eigenvalue of a graph,  $\tilde{\lambda}_2(G) = \max\{\lambda \mid \lambda \text{ an eigenvalue of } A_G \text{ and } \lambda \neq d\}$ , the two-sided Cheeger inequality reads as follows:  $\frac{d - \tilde{\lambda}_2(G)}{2} \leq h(G) \leq \sqrt{2d \cdot (d - \tilde{\lambda}_2(G))}$ . Thus, this weaker notion of spectral expansion is, in fact, equivalent to combinatorial expansion.

### 1.2.2 Topological Overlapping

In [G], Gromov considered a slightly weaker but essentially equivalent definition of combinatorial high dimensional expansion, which we term here cosystolic expansion (see below). The motivation of [G] was the study of a fiber-wise complexity of embedding simplicial complexes into Euclidean spaces; A property that generalizes the embedding complexity of a graph. More specifically, Gromov [G] considered the following topological overlapping property.

**Definition 1.2** (Topological overlapping property, TOP). *A  $d$ -dimensional simplicial complex  $X$  has the  $\mu$ -topological overlapping property for  $\mu > 0$ , if for every continuous map  $f: X \rightarrow \mathbb{R}^d$ , there exists a point  $p \in \mathbb{R}^d$  whose preimage,  $f^{-1}(p)$ , intersects at least  $\mu$ -fraction of the  $d$ -simplices of  $X$ , i.e.,*

$$\frac{|\{\sigma \in X(d) : p \in f(\sigma)\}|}{|X(d)|} \geq \mu \quad (1.1)$$

*A family of complexes is said to have the topological overlapping property (TOP), if each member of the family has the  $\mu$ -topological overlapping property, for the same  $\mu > 0$ .*

Gromov proved the following criterion for possessing the topological overlapping property (see [DKW] for a detailed proof).

**Theorem 1.3** (Gromov’s TOP criterion-*Informal*, for formal see Theorem 6.6). *Cosystolic expanders have the topological overlapping property.*

This result enabled Gromov to prove that the complete complexes have the topological overlapping property (a claim which is by no means obvious or even intuitive). Gromov then raised the following question:

**Question 1.4** (Gromov). *Do there exist bounded degree complexes with the topological overlapping property, in every dimension?*

In a recent breakthrough, [KKL] presented the first bounded degree cosystolic expanders, which imply by Gromov’s work, the first bounded degree complexes with the topological overlapping property. However, the work of [KKL] applies only for dimension two.

### 1.2.3 Property Testing

As noted in [KL], combinatorial expansion of graphs can be thought of as a *property testing* question, where the property is being a non-expanding set, namely a member of  $\{A \subset V \mid |\delta(A)| = 0\}$ , and the  $\epsilon$ -expansion requirement requires that for sets  $S \subseteq V$  that are not in the property (i.e., that are not “non-expanding”) the number of violated tests (i.e. number of edges in  $\delta(S)$ ) is proportional to the distance from the property. This relation between combinatorial expansion and property testing, carries also to higher dimensional definition of combinatorial expansion that is discussed in the following section.

### 1.2.4 On Spectral and Combinatorial Expansion in Higher Dimensions

As mentioned earlier, a very useful fact in expander graph theory, is that spectral expansion implies combinatorial expansion. Unfortunately, it was shown in [GW] that in higher dimensional simplicial complexes, this is no longer true (the spectral expansion is measured with respect to high order laplacians; we omit the definition here, see [PRT]). Namely, high dimensional spectral expanders need not be high dimensional combinatorial expanders (neither cosystolic

nor coboundary expanders), and in fact, no spectral parameter of a complex is currently known to control its combinatorial expansion.

On the other hand, in [PRT], [GS], [Par], [Ros], [Opp] and [FGLNP], certain high dimensional generalizations of the expander mixing lemma and the Cheeger inequality were proven. These results show that the spectral gap of the higher order Laplacians controls some of the pseudorandomness behavior of a high dimensional complex.

### 1.3 High Dimensional Expanders - Definitions

Let us now define what does it mean to be a combinatorial high dimensional expander. Following, we discuss two commonly studied generalizations of the notion of combinatorial expansion to higher dimensions, the coboundary expansion (due to Linial-Meshulam), and the cosystolic expansion (due to Gromov).

#### 1.3.1 Expander Graphs as Simplicial Complexes

Let us review again what does it means for a 1-dimensional simplicial complex (i.e. a graph),  $X = (V, E)$ , to be an  $\epsilon$ -combinatorial expander. A subset of vertices  $S \subset V$  can be thought of as a function  $S : V \rightarrow \mathbb{F}_2$ . The boundary of a subset  $S \subset V$  is defined as follows:

$$\delta(S) = \{(v_1, v_2) \in E \mid S(v_1) + S(v_2) \not\equiv 0 \pmod{2}\} = E(S, \bar{S}).$$

A subset  $S \subset V$  is called *non-expanding* if  $|\delta(S)| = 0$ , otherwise it is called *expanding*. Note that a graph is connected if and only if the only non-expanding sets of the graph are  $\emptyset$  and  $V$ . We call these non-expanding sets *trivial*. A graph  $X = (V, E)$  is an  $\epsilon$ -combinatorial expander if and only if the following two properties holds:

- Every non-expanding set of vertices is trivial, i.e. it is either  $V$  or  $\emptyset$ .
- Every set of vertices that expands, must expand with proportion to its distance from the non-expanding sets, i.e.

$$\forall S \subset V, S \notin \{\emptyset, V\}, \quad \frac{|\delta(S)|}{\text{dist}(S, \{W \subseteq V \mid |\delta(W)| = 0\})} = \frac{|E(S, \bar{S})|}{\min\{|S|, |\bar{S}|\}} \geq \epsilon.$$

#### 1.3.2 Triangle Complex Expanders

In order to demonstrate the generalizations of the combinatorial expansion of graphs to higher dimensions we begin with 2-dimensional simplicial complexes, namely triangle complexes,  $X = (V, E, T)$ . Note that the 1-skeleton of  $X$  (denoted  $X^{(1)} = (V, E)$ ) is the graph obtained from  $X$  by "forgetting" its triangles.  $X$  is called an  $\epsilon$ -vertex coboundary expander if its 1-skeleton is an  $\epsilon$ -combinatorial expander graph.

A set of edges,  $S \subseteq E$ , can be thought of as a function  $S : E \rightarrow \{0, 1\}$ . The boundary of a subset  $S \subseteq E$  is defined as follows:

$$\delta(S) := \{(v_1, v_2, v_3) \in T \mid S((v_1, v_2)) + S((v_1, v_3)) + S((v_2, v_3)) \not\equiv 0 \pmod{2}\}.$$

A subset  $S \subseteq E$  is called *non-expanding* if  $|\delta(S)| = 0$ , otherwise  $S$  is called *expanding*.

If we divide the set of vertices of the complex into two parts and consider all the edges that cross between parts, then such a set of edges is called a *cut*. If  $S \subseteq E$  is a cut then  $S$  is non-expanding, namely sets of edges that correspond to cuts are always non expanding; These sets of edges are called the *trivial* non-expanding sets. In some complexes there could be other sets of edges that will not expand, besides the trivial ones.

**Definition 1.5** (Coboundary expansion-*Informal*, for formal see §2.2). *A triangle complex is an  $\epsilon$ -edge coboundary expander if it satisfies the following:*

- Every non-expanding set of edges is trivial, i.e. it is a cut.
- Every set of edges that expands, must expand with proportion to its distance from the non-expanding sets, i.e.

$$\forall S \subset E, S \notin \{\text{cuts}\}, \quad \frac{|\delta(S)|}{\text{dist}(S, \{W \subseteq E \mid |\delta(W)| = 0\})} \geq \epsilon.$$

*A triangle complex is an  $\epsilon$ -coboundary expander, if it expands both with respect to vertices and edges, namely if it is both an  $\epsilon$ -vertex coboundary expander and an  $\epsilon$ -edge coboundary expander.*

The other generalization of combinatorial expansion is the following.

**Definition 1.6** (Cosystolic expansion-*Informal*, for formal see §2.2). *Same as coboundary expansion with the relaxation that sets of edges that do not expand are either trivial (as in the coboundary expansion case) or large.*

**Remark 1.7.** *Note that coboundary expansion implies cosystolic expansion, but not vice versa.*

**Remark 1.8.** *A complex in which sets of edges that do not expand must be either trivial or large, is said to have a large cosystole property (see Definition 2.14).*

### 1.3.3 High Dimensional Expanders

The above definitions of expansion in triangle complexes can be generalized to higher dimensions, with the requirements that the complex expands with respect to higher order cells (i.e. triangles, pyramids, etc), and the expansion is measured by the coboundary map.

Following we review some terminology of simplicial complexes that would allow us to state the definitions of high dimensional combinatorial expansion, in somewhat formal way. A simplicial complex  $X$  over a set of vertices  $V$  is a collection of subsets of  $V$  which is closed under taking subsets, namely, if  $\tau \in X$  and  $\sigma \subset \tau$  then  $\sigma \in X$ . The members of  $X$  are called faces. For a face  $\sigma \in X$ , its dimension is defined as  $\dim(\sigma) = |\sigma| - 1$ , and the dimension of the complex is defined as  $\dim(X) = \max_{\sigma \in X} \dim(\sigma)$ . For any  $k \leq \dim(X)$ , denote by  $X(k)$  the collection of  $k$ -dimensional faces ( $k$ -faces) in  $X$ , i.e.,  $X(0)$  is the collection of vertices of  $X$ ,  $X(1)$  is the collection of edges of  $X$ ,  $X(2)$  is the collection of triangles of  $X$ , etc...

A set of  $k$ -faces,  $S \subseteq X(k)$ , can be thought of as a function  $S : X(k) \rightarrow \{0, 1\}$  (such functions are called  $k$ -cochains). The (co)boundary of a subset  $S \subset X(k)$  is defined as follows:

$$\delta(S) := \{\tau = (v_0, \dots, v_{k+1}) \in X(k+1) \mid \sum_{i=0}^{k+1} S(\tau \setminus \{v_i\}) \not\equiv 0 \pmod{2}\}.$$

The *non-expanding* sets,  $S \subset X(k)$ , are called the  $k$ -cocycles, these are by definition the sets with zero coboundary, i.e.  $|\delta(S)| = 0$ . The *trivial non-expanding* sets,  $S \subset X(k)$ , are called the  $k$ -coboundaries,  $S$  is a  $k$ -coboundary if there exists some  $T \subset X(k-1)$  such that  $S = \delta(T)$  (Note that a cut is actually a coboundary of some set of vertices).

A complex is said to be an  $\epsilon$ -coboundary (resp.  $\epsilon$ -cosystolic) expander, if the following holds:

- Every non-expanding set of  $k$ -faces ( $0 \leq k < \dim(X)$ ) is trivial (resp. or large).

- Every set of  $k$ -faces ( $0 \leq k < \dim(X)$ ) that expands, must expand with proportion to its distance from the non-expanding sets, i.e.

$$\forall S \subset X(k), |\delta(S)| \neq 0, \quad \frac{|\delta(S)|}{\text{dist}(S, \{W \subseteq X(k) \mid |\delta(W)| = 0\})} \geq \epsilon.$$

## 1.4 Our Contribution

In this work we show, for the first time, a local to global criterion on a complex that implies cosystolic expansion. This criterion allows us to present the first explicit bounded degree cosystolic expanders, in every dimension. These bounded degree cosystolic expanders imply bounded degree complexes with the topological overlapping property. Thus, we solve affirmatively an open question raised by Gromov, who asked whether bounded degree complexes with the topological overlapping property could at all exist (see Question 1.4 above).

### 1.4.1 A Criterion for Cosystolic Expansion

The bulk of this work is devoted to proving a local to global criterion for cosystolic expansion. Prior to introducing this criterion, we recall the definitions of *skeletons* and *links* of a  $d$ -dimensional simplicial complex  $X$ . For  $k \leq d$ , the  $k$ -dimensional *skeleton* of  $X$ , denoted  $X^{(k)}$ , is the complex obtained by deleting from  $X$  all faces of dimension greater than  $k$ . E.g. the 1-dimensional skeleton of a complex, is its underlying graph. For  $\emptyset \neq \sigma \in X$ , the *link* of the face  $\sigma$  in  $X$ , denoted  $X_\sigma$ , is the complex obtained by picking only the faces in  $X$  that contain  $\sigma$ , and removing  $\sigma$  from all these faces. Intuitively, a link is a discrete analogue of the notion of a unit sphere in a simplicial complex. One can similarly define the link of the empty-set face,  $\emptyset$ , but this turns out to be everything, i.e.  $X_\emptyset = X$ .

We are now ready to present our first main theorem, which is a criterion for cosystolic expansion:

**Theorem 1.9** ((Main-I) Criterion for cosystolic expansion - Informal, for formal see Theorem 4.1). *Let  $X$  be a bounded degree  $d$ -dimensional complex which satisfies the following:*

- *Each link of  $X$  is a coboundary expander.*
- *The underlying graphs of  $X$ , and of all of its links, are excellent expander graphs (see below).*

*Then the  $(d - 1)$ -dimensional skeleton of  $X$  is a cosystolic expander.*

Using this criterion we obtain the first bounded degree complexes that are cosystolic expanders.

### 1.4.2 Bounded Degree Cosystolic Expanders and Topological Overlapping

The well known Ramanujan complexes (see [L2]) are bounded degree high dimensional complexes. Ramanujan complexes with sufficiently high degree (i.e. sufficient thickness with comparison to their dimension) satisfy the requirements of Theorem 1.9. The fact that the 1-dimensional skeletons of these complexes are excellent spectral expanders follows from their Ramanujaness. As their links are spherical buildings, it follows from the work of Gromov [G] (see also [LMM]) that the links are coboundary expanders. The fact that the links (i.e. the spherical buildings) are excellent spectral expanders is proven in this work (see Theorem 1.12 below). Thus, applying our expansion criterion (Theorem 1.9) to the Ramanujan complexes yield the first bounded degree cosystolic expanders. This forms our second main theorem.

**Theorem 1.10** ((Main-II) Bounded degree cosystolic expanders-*Informal*, for formal see Corollary 6.5). *For any  $d \in \mathbb{N}$  and  $q = q(d) \gg 0$  large enough, the  $d$ -dimensional skeletons of the  $(d+1)$ -dimensional  $q$ -regular Ramanujan complexes of [LSV1], form a family of bounded degree  $d$ -dimensional complexes which are cosystolic expanders.*

Combining the above construction of bounded degree cosystolic expanders (Theorem 1.10), with Gromov's TOP criterion (Theorem 1.3), implies an affirmative answer to Gromov's question 1.4, concerning the existence of bounded degree complexes with the topological overlapping property.

**Corollary 1.11** (Bounded degree complexes with TOP-*Informal*, for formal see Corollary 6.7). *For any  $d \in \mathbb{N}$  and  $q = q(d) \gg 0$  large enough, the  $d$ -dimensional skeletons of the  $(d+1)$ -dimensional  $q$ -regular Ramanujan complexes of [LSV1], form a family of bounded degree  $d$ -dimensional complexes with the topological overlapping property (TOP).*

Note that by applying to the work of [LSV2], which presented an explicit construction of Ramanujan complexes, one gets an explicit construction of bounded degree complexes which are both cosystolic expanders and posses the topological overlapping property.

### 1.4.3 Spherical Buildings are Excellent Spectral Expanders

As discussed in Section 1.4.2, the missing piece required for applying our criterion for cosystolic expansion (Theorem 1.9) to the Ramanujan complexes, is a statement that spherical buildings, that form the links of the Ramanujan complexes, are excellent spectral expanders. This exact statement is our third result.

**Theorem 1.12** (Spherical buildings are excellent expanders-*Informal*, for formal see Theorem 5.18). *The underlying graphs of the spherical buildings, whose thickness is sufficiently large (in comparison with their dimension), are excellent spectral expanders.*

Spherical buildings are complexes which display a rich amount of symmetry as well as a rigid geometric structure. An example of a spherical building is the complex whose vertices are the non-trivial linear subspaces of  $\mathbb{F}_q^d$ , and whose faces are the flags of subspaces. We show that a graph with strong enough symmetric and geometric properties, like the spherical buildings, is an excellent spectral expander (see §5.2). Consequentially, we get that the spherical buildings satisfy a form of a regularity Lemma: There exists a partition of their vertices into constant many parts, such that the sets of edges between every two different parts are pseudorandom. Note that spherical buildings yield families of complexes whose degrees are neither dense, nor bounded.

**Remark 1.13.** *After a completion of our paper, Izhar Oppenheim has pointed to us that a similar result to Theorem 1.12, could potentially be deduced from the work of [Opp], as a special case of Theorem 8.12 there. Our proof that exploits the geometric structure of the spherical buildings, allows us to get a proof which is significantly shorter.*

### 1.4.4 Applications to Error Correcting Codes

In the following we derive linear lower bound on the cosystole obtained from the Ramanujan complexes. Such a linear lower bound has implications to error correcting codes that we discuss next. Recall that a cosystole is the minimal size of a non-expanding set (cocycle) in a complex, which is not trivial (coboundary). An immediate consequence of Theorem 1.9, is the following linear lower bound on the cosystole obtained from the Ramanujan complexes.



**Corollary 1.14** (Large cosystoles, Informal, for formal see Corollary 6.5). *Let  $X$  be a complex which satisfies the conditions of Theorem 1.9. Then any cosystole of  $X$ , is of linear size in  $|X|$ . In particular, any cosystole of a Ramanujan complex  $X$ , is of linear size in  $|X|$ .*

Following we mention a few applications of our work, to error correcting codes (both classical and quantum).

**Locally testable codes.** The non-expanding sets of a cosystolic expander form a locally testable code. The tester of such a code is the cocycle-tester defined in [KL, § 3]. However, the locally testable code obtained has poor distance (since two codewords can differ on the link of a single face).

**Quotient codes.** The cosystolic bound (Corollary 1.14) implies (non-linear) codes with good distance. The codewords of such a non-linear code are non-expanding sets whose sum is a non-trivial non-expanding set. Namely, the codewords of this code are representatives of the different cohomology classes of the complex. The linear lower bound on the cosystole implies linear distance of this code.

**Quantum error correcting code.** A common way to obtain quantum error correcting codes is via homological codes. The distance of a homological code is the minimum between the systole and cosystole of the corresponding complex (see [GL]). Thus, the above linear cosystolic bound (Corollary 1.14) could be considered as a step towards obtaining a quantum error correcting codes from the Ramanujan complexes.

## 1.5 Road Map of our Proof

In order to prove Theorem 1.9, we use a reduction that was introduced in [KKL], showing that for obtaining Theorem 1.9, it is enough to show the following isoperimetric inequality. (See §4.3 below for a detailed explanation of this reduction).

**Theorem 1.15** (Isoperimetric inequality-Informal, for formal see Theorem 4.3). *Let  $X$  be a  $d$ -dimensional complex (possibly of unbounded degree!) satisfying the conditions of Theorem 1.9; There exist constants,  $\mu, \epsilon > 0$  (independent of the size and the degree of  $X$ ), such that: If  $A$  is a locally minimal  $k$ -cochain  $k < d$  and  $\|A\| \leq \mu$ , then  $\frac{\|\delta(A)\|}{\|A\|} > \epsilon$ .*

*The norm  $\|A\|$  (see §2) could be thought of as the normalized support size of  $A$ . By locally minimal (see §2) we mean that the norm of  $\|A\|$  could not be reduced by adding to it some trivial non-expanding set.*

To sketch the proof of Theorem 1.15, we need the following definition.

**Definition 1.16** (Fat faces-Informal, for formal see Definition 4.5). *For a fixed  $k$ -cochain,  $A$ , define the following: A  $k$ -face is called fat if it belongs to  $A$ . A  $(k-1)$ -face is called fat if it is contained in "many" fat  $k$ -faces. A  $(k-2)$ -face is called fat if it is contained in "many" fat  $(k-1)$ -faces, etc...*

The reason for the definition of a fat face is the following. Usually, for a pair of  $i$ -fat faces that intersects on an  $(i-1)$ -face ( $i \leq k$ ), their intersection is a fat  $(i-1)$ -face. This is going to play a major role in our proof. Following we describe the simplest instance of this phenomena. Let  $A$  be a  $k$ -cochain,  $\sigma$  a non-fat  $(k-1)$ -face, and consider the 1-skeleton of the link of  $\sigma$ ,  $G = X_\sigma^{(1)}$ . Since  $\sigma$  is non-fat, the elements of  $A$  which sit on  $\sigma$ ,  $A_\sigma$ , is a small set of vertices in  $G$ . So, assuming  $G$  is a good expander, by the mixing lemma there are *very few* edges in  $G$  (which are  $(k+1)$ -faces in  $X$ ), with two vertices in  $A_\sigma$ . This means that almost all of the  $(k+1)$  faces that contain a pair of  $k$  faces that intersect on  $\sigma$ , have at least one non-fat  $k$ -face in the pair. Thus, in almost all of the  $(k+1)$  faces that contain a pair of *fat*  $k$ -faces that intersect on a  $(k-1)$ -face  $\sigma$  it must be that  $\sigma$  is fat. This situation holds in every dimensions.

**Lemma 1.17** (Fat Mixing Lemma-Informal, for formal see Proposition 4.9). *Let  $X$  be a complex whose 1-dimensional skeleton and the 1-dimensional skeletons of each of its links, are excellent spectral expanders. Then, for every two fat  $i$ -faces, whose union is a  $(i+1)$ -face, their intersection, which is a  $(i-1)$ -face, is "usually" fat.*

Actually, we will need to show that the same result holds for any two faces which may *not* be of the same dimension. Namely, we wish to show that the intersection of a fat  $k$ -face  $t$  with a fat  $i$ -face  $\sigma$  that participate together in a  $k+1$ -face  $\Delta$  is a fat  $i-1$  face. However by Lemma 1.17 we can get that only for  $t$  and  $\sigma$  which are both  $i$ -faces.

The way to overcome this is by introducing a ladder: Namely we will consider *special* fat  $i$ -faces:  $\sigma$  will be called a special fat  $i$ -face that participates together with  $t$  in a  $k+1$  face  $\Delta$  if  $\sigma$  is contained in an  $i+1$  fat face  $\sigma_{i+1}$  that is contained in a fat  $i+2$  face  $\sigma_{i+2}$  that is contained .... in a fat  $k$  face  $\sigma_k$  that is contained in  $\Delta$ . In this case we say that  $\sigma = \sigma_i$  has a *ladder* (with which it can climb to a  $k$ -fat face in the  $k+1$  face  $\Delta$ ).

Now  $\sigma_k$  and  $t$  are both  $k$ -fat faces, thus, by Lemma 1.17 their intersection is mostly fat (if not we are in a negligible case that contributes to our error term), so  $t \cap \sigma_k$  is a  $k-1$  fat face, moreover this  $k-1$  fat face has intersection of size  $k-2$  with  $\sigma_{k-1}$ , hence by Lemma 1.17 again  $(t \cap \sigma_k) \cap \sigma_{k-1} = t \cap \sigma_{k-1}$  is a fat  $k-2$  face.

Now again:  $t \cap \sigma_{k-1}$  is a fat  $k-2$  face, moreover this  $k-2$  fat face has intersection of size  $k-3$  with  $\sigma_{k-2}$ , hence by Lemma 1.17 again,  $(t \cap \sigma_{k-1}) \cap \sigma_{k-2} = t \cap \sigma_{k-2}$  is a fat  $k-3$  face.

Repeating it again and again we get that:  $t \cap \sigma_{i+1}$  is a fat  $i$  face, moreover this  $i$  fat face has intersection of size  $i-1$  with  $\sigma = \sigma_i$ , hence by Lemma 1.17 again,  $(t \cap \sigma_{i+1}) \cap \sigma_i = t \cap \sigma_i$  is a fat  $i-1$  face as we wished to conclude. Thus we get:

**Lemma 1.18** (Ladders Lemma-Informal, for formal see Lemma 4.11). *Let  $X$  be a complex whose 1-dimensional skeleton and the 1-dimensional skeletons of each of its links, are excellent spectral expanders. Then, for every fat  $i$ -face (with a ladder!) and a fat  $k$ -face, whose union is a  $(k+1)$ -face,  $i \leq k$ , their intersection, which is a  $(i-1)$ -face, is "usually" fat (with a ladder).*

Now consider the following situation: Let  $A$  be a locally minimal  $k$ -cochain such that a constant fraction of its faces have a fat (ladder)  $i$ -face. Since we assume that all the links are coboundary expanders we get that  $A_\sigma$  has a large coboundary in the link of these fat  $i$ -faces  $\sigma$ . Now, essentially one of the following two possibilities happens, either these local coboundaries are in fact global coboundaries, and hence  $A$  has a large coboundary, or in many of the local coboundaries there is another fat  $k$ -face,  $t \in A$ , which is "unseen" from  $\sigma$  (i.e.  $t$  does not contain  $\sigma$ ); namely, such local coboundary does not contribute to the global coboundary of  $A$ . Thus, by Lemma 1.18,  $t \cap \sigma$  is a fat  $(i-1)$ -face. This implies the following lemma.

**Lemma 1.19** (Seeping Lemma-Informal, for formal see Proposition 4.12). *Let  $X$  be a complex which satisfies the conditions of Theorem 1.15. Let  $A$  be a locally minimal cochain, and assume that a constant fraction of the faces in  $A$  contain a fat  $i$ -face. Then essentially one of the following holds:*

- The coboundary of  $A$  is large, i.e.  $\frac{\|\delta(A)\|}{\|A\|} \geq \epsilon$ .
- A constant fraction of the faces in  $A$  contain a fat  $(i-1)$ -face.

Finally, we can derive Theorem 1.15, from the Lemma 1.19. Let  $A$  be a locally minimal  $k$ -cochain in  $X$ , with a small norm. Assume in contradiction that  $\frac{\|\delta(A)\|}{\|A\|} < \epsilon$ , then by iterative application of Lemma 1.19, we get that a fraction of the faces of  $A$  contains a fat  $(-1)$ -face. However this is impossible, since the only  $(-1)$ -face,  $\emptyset$ , is non-fat due to the fact that  $A$  has a

small norm (if  $\emptyset$  was fat, then there are many fat vertices, on them sits many fat edges, ..., on them sits many fat  $k$ -faces, which forces  $A$  to have a large norm). This completes the proof of Theorem 1.15.

**Remark 1.20.** *In retrospect we can view the proof strategy in [KKL] as saying that for a  $d$ -dimensional complex  $X$ , and a  $k$ -cochain  $A$ ,  $k < d$ ; If  $A$  has many  $k - 1$  faces which are fat, then  $A$  has many 0-faces which are fat. This turns to be true for dimension  $k = 3$ , but it stops to be correct for dimension  $k > 3$ .*

## 1.6 Organization of the Paper

In section 2 we review some basic definition of simplicial complexes, links, norms and high dimensional expansion. In section 3 we introduce a mixing lemma for partite graphs. In section 4 we prove the main theorems, the criteria for an isoperimetric inequality and a cosystolic expansion. In section 5 we show that the spherical buildings satisfy the spectral condition which implies a good mixing property on their skeletons. Finally, in section 6, we combine all previous results to show that the Ramanujan complexes satisfies the necessary conditions to imply the isoperimetric inequality, and we derive from them explicit bounded degree cosystolic expanders of every dimension.

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# 2 Preliminaries on Complexes and Expansions

In this section we present the basic definitions and properties of simplicial complexes with norms, as well as notions of high-dimensional expansions.

## 2.1 Complexes, Norms and Links

Throughout this paper we shall use the following notations:

A simplicial complex,  $X$ , with a set of vertices  $V$ , is family of subsets of  $V$ ,  $X \subset 2^V$ , which is closed under inclusions, i.e. if  $F \in X$  and  $E \subset F$  then  $E \in X$  (note that the empty-set is always a face in any complex). Call the elements of  $X$ , faces or simplices. The dimension of a simplex  $F \in X$ , is defined as  $\dim(F) = |F| - 1$ , and the dimension of the entire complex is defined as the maximal dimension of a simplex in it,  $\dim(X) = \max_{F \in X} \dim(F)$ . A complex is said to be pure if all its maximal faces are of the same dimension.

For convenience sake, by a  $k$ -face we mean a  $k$ -dimensional face of the complex, and by a  $d$ -complex we will always mean a finite  $d$ -dimensional pure simplicial complex.

Let  $X$  be a  $d$ -complex. For any  $-1 \leq k \leq d$ , denote by  $X(k)$  the collection of  $k$ -faces in  $X$ , and by  $C^k = C^k(X; \mathbb{F}_2) = \{f : X(k) \rightarrow \mathbb{F}_2\} \cong \{A \subset X(k)\}$  the space of  $k$ -cochains. As usual, denote by  $\delta = \delta^k : C^k \rightarrow C^{k+1}$  the  $k$ -coboundary map, and by  $B^k = B^k(X, \mathbb{F}_2) = \text{Im}(\delta^{k-1})$  and  $Z^k = Z^k(X, \mathbb{F}_2) = \ker(\delta^k)$  the spaces of  $k$ -coboundries and  $k$ -cocycles. Recall that  $\delta \circ \delta = 0$ , so  $B^k \subset Z^k$ .

**Definition 2.1** (Norm). *Define the following norm of the space of cochains:*

$$\|\cdot\| = \|\cdot\|^k : C^k \rightarrow [0, 1], \quad \|A\| = \sum_{\sigma \in A} w(\sigma) \quad \text{where} \quad w(\sigma) = \frac{|\{F \in X(d) \mid \sigma \subset F\}|}{\binom{d+1}{|\sigma|} \cdot |X(d)|}$$

Note the following two basic properties of the norm which we shall use freely. For any  $-1 \leq k \leq d$ , and any  $A, B \in C^k$  (i.e.  $A, B \subset X(k)$ ), then:

- 1)  $\|A\| = 0$  if and only if  $A = \emptyset$ , and  $\|A\| = 1$  if and only if  $A = X(k)$ .
- 2)  $\|A \cup B\| \leq \|A\| + \|B\|$  with equality if and only if  $A$  and  $B$  are disjoint.

**Definition 2.2** (Container). *Let  $X$  be a  $d$ -complex,  $-1 \leq k \leq r \leq d$ . For any  $A \in C^k$ , define  $\Gamma^r(A) \in C^r$  to be the following*

$$\Gamma^r(A) = \{F \in X(r) \mid \exists \sigma \in A, \sigma \subset F\}.$$

**Lemma 2.3** (Container Lemma). *Let  $X$  be a  $d$ -complex,  $-1 \leq k \leq r \leq d$ . Then for any  $A \in C^k$ ,*

$$\|A\| \leq \|\Gamma^r(A)\| \leq \binom{r+1}{k+1} \cdot \|A\| \quad (2.1)$$

*Proof.* Each  $F \in \Gamma^r(A)$  contains between 1 and  $\binom{r+1}{k+1}$  faces from  $A$ . □

**Definition 2.4** (Link). *Let  $X$  be a  $d$ -complex, and let  $\sigma \in X$  be any face.*

*The link of  $\sigma$  in  $X$  is defined as the the following  $(d - |\sigma|)$ -complex,*

$$X_\sigma = \{\tau \in X \mid \sigma \sqcup \tau \in X\}$$

*For any  $-1 \leq k \leq d - |\sigma|$ , denote by  $C_\sigma^k = C^k(X_\sigma, \mathbb{F}_2)$  the space of  $k$ -cochains of  $X_\sigma$ , by  $\|\cdot\|_\sigma = \|\cdot\|_\sigma^k : C_\sigma^k \rightarrow [0, 1]$  the norm associated to the complex  $X_\sigma$ , and by  $\delta_\sigma = \delta_\sigma^k : C_\sigma^k \rightarrow C_\sigma^{k+1}$  the  $k$ -coboundary map.*

**Definition 2.5** (Localization and lifting). *Let  $X$  be a  $d$ -complex, and let  $\sigma \in X$  be any face. Define the following maps between the original complex,  $X$ , and the link complex  $X_\sigma$ :*

- *The first map, called the localization (w.r.t.  $\sigma$ ), takes a cochain of the original complex, and restrict only to the faces which contains  $\sigma$ , and then delete  $\sigma$  from each of them, producing a cochain for the link. Concretely,*

$$\square_\sigma : C^* \rightarrow C_\sigma^{*-|\sigma|}, \quad A_\sigma := \{\tau \in X_\sigma \mid \tau \sqcup \sigma \in A\}.$$

- *The second map, called the lifting (w.r.t.  $\sigma$ ), takes a cochain of the link complex, and adds  $\sigma$  to each face in it, producing a cochain of the original complex. Concretely,*

$$\tilde{\square} : C_\sigma^* \rightarrow C^{*+|\sigma|}, \quad \tilde{A} := \{\tau \sqcup \sigma \in X \mid \tau \in A\}.$$

The connection between the global norm (of  $X$ ) and the link norm (of  $X_\sigma$ ) is described in the following Lemma.

**Lemma 2.6** (Global-to-local Lemma). *Let  $X$  be a  $d$ -complex and let  $\sigma \in X$ . For any  $0 \leq k \leq d - |\sigma|$ , and any  $A \in C_\sigma^k$ , then,*

$$\|\tilde{A}\| = \binom{|\sigma| + k}{k} \cdot w(\sigma) \cdot \|A\|_\sigma \quad (2.2)$$

*Proof.* Since the norm of a cochain is define by extending linearly the weight function, it is suffice to show the claim for  $A$  which is a singletons. Denote by  $w = w_X$  be the weight function of the original complex, and  $w_\sigma = w_{X_\sigma}$  the weight function of the link. In the language of links, the weight norm is interpreted as  $w(\tau) = \frac{|X_\tau(d-|\tau|)|}{\binom{d+1}{|\tau|} \cdot |X(d)|}$  (and similarly for  $w_\sigma$ ). So, for any  $\tau \in X_\sigma(k)$  and  $A = \{\tau\} \in C_\sigma^k$ , we have

$$\begin{aligned} w(\sigma) \cdot \|A\|_\sigma &= w(\sigma) \cdot w_\sigma(\tau) = \frac{|X_\sigma(d-|\sigma|)|}{\binom{d+1}{|\sigma|} \cdot |X(d)|} \cdot \frac{|(X_\sigma)_\tau(d-|\sigma|-|\tau|)|}{\binom{d-|\sigma|+1}{|\tau|} \cdot |X_\sigma(d-|\sigma|)|} \\ &= \frac{|X_\sigma(d-|\sigma|)|}{\binom{d+1}{|\sigma|} \cdot |X(d)|} \cdot \frac{|X_{\sigma \cup \tau}(d-|\sigma \cup \tau|)|}{\binom{d-|\sigma|+1}{|\tau|} \cdot |X_\sigma(d-|\sigma|)|} = \frac{|X_{\sigma \cup \tau}(d-|\sigma \cup \tau|)|}{\binom{d+1}{|\sigma|} \cdot \binom{d-|\sigma|+1}{|\tau|} \cdot |X(d)|} \\ &= \frac{\binom{d+1}{|\sigma \cup \tau|}}{\binom{d+1}{|\sigma|} \cdot \binom{d-|\sigma|+1}{|\tau|}} \cdot w(\sigma \cup \tau) = \frac{1}{\binom{|\sigma|+|\tau|}{|\tau|}} \cdot \|\tilde{A}\|. \end{aligned} \quad (2.3)$$

which finishes the proof.  $\square$

The following Lemma, shows how all the localization together determines the original cochain (at least norm-wise).

**Lemma 2.7.** *Let  $X$  be a  $d$ -complex,  $0 \leq j \leq k \leq d$ , and let  $A \in C^k$ . Then*

$$\binom{k+1}{j+1} \cdot \|A\| = \sum_{\sigma \in X(j)} \|\tilde{A}_\sigma\| \quad (2.4)$$

*Proof.* By the definition of the norm and Fubini's Theorem,

$$\sum_{\sigma \in X(j)} \|\tilde{A}_\sigma\| = \sum_{\sigma} \sum_{\sigma \subset \tau \in A} w(\tau) = \sum_{\tau \in A} \sum_{\sigma \subset \tau} w(\tau) = \sum_{\tau \in A} \binom{k+1}{j+1} w(\tau) = \binom{k+1}{j+1} \|A\|$$

$\square$

Let us now introduce a notion that will serve us when talking on expansion.

**Definition 2.8** (Minimal and locally minimal). *A cochain  $A \in C^k$  is said to be minimal if*

$$\|A\| = \min_{b \in B^k(X, \mathbb{F}_2)} \|A + b\|.$$

*A cochain  $A \in C^k$  is said to be locally minimal if for any  $\emptyset \neq \sigma \in X$ , the localization of  $A$  w.r.t.  $\sigma$ ,  $A_\sigma$ , is a minimal cochain in the link  $X_\sigma$ .*

**Lemma 2.9.** *If  $A$  is minimal cochain, and  $A' \subset A$  is a sub-cochain, then  $A'$  is also a minimal cochain.*

*Proof.* First note that, since  $A \setminus A'$  and  $A'$  are disjoint,  $\|A\| = \|A \setminus A'\| + \|A'\|$ . Next note that, since the sum of two cochains is equal to their symmetric difference, for any cochain  $c \in C^k$ ,

$$\begin{aligned} (A + c) \setminus (A' + c) &= ((A \setminus c) \cup (c \setminus A)) \setminus ((A' \setminus c) \cup (c \setminus A')) \subset \\ &\subset ((A \setminus c) \setminus (A' \setminus c)) \cup ((c \setminus A) \setminus (c \setminus A')) = ((A \setminus c) \setminus (A' \setminus c)) \subset A \setminus A' \end{aligned} \quad (2.5)$$

where in the second to last step, the equality follows from the fact that  $A' \subset A$ . So, combining this with the triangle inequality,

$$\|A + c\| - \|A' + c\| \leq \|(A + c) \setminus (A' + c)\| \leq \|A \setminus A'\| = \|A\| - \|A'\| \quad (2.6)$$

Now, if  $c \in B^k$  is a coboundary, then by the minimality of  $A$  we get,

$$\|A'\| = \|A'\| - \|A\| + \|A\| \leq \|A'\| - \|A\| + \|A + c\| \leq \|A' + c\| \quad (2.7)$$

where the last inequality is (2.6), which finishes the proof.  $\square$

## 2.2 High-Dimensional Expansion

Here we present several definitions of expansion for simplicial complexes.

**Definition 2.10** (Coboundary and cocycle expansion). *Let  $X$  be a  $d$ -complex and  $0 \leq k < d$ . Define the  $k$ -dimensional coboundary expansion parameter of  $X$  to be:*

$$\text{Exp}_b^k(X) = \min\left\{\frac{\|\delta(A)\|}{\min_{b \in B^k} \|A + b\|} \mid A \in C^k \setminus B^k\right\}. \quad (2.8)$$

*Define the  $k$ -dimensional cocycle expansion parameter of  $X$  to be:*

$$\text{Exp}_z^k(X) = \min\left\{\frac{\|\delta(A)\|}{\min_{z \in Z^k} \|A + z\|} \mid A \in C^k \setminus Z^k\right\}. \quad (2.9)$$

**Remark 2.11.** *Before moving to the definitions of high dimensional expanders, let us spell out what both expansion parameters says in the special case of graphs. In the graph case  $d = 1$  and  $k = 0$ , the coboundaries are just  $V$  and  $\emptyset$ , and the cocycles are the unions of connected components of the graph.*

*The coboundary expansion parameter is equal to the Cheeger constant of the entire graph, while the cocycle expansion parameter is equal to the minimum of the Cheeger constants in each connected component of the graph.*

*So, a large cocycle expansion parameter imply that each connected component of the graph is a good expander on its own, however the graph itself can be disconnected, in particular not an expander.*

We now present the first definition of high dimensional expanders, the coboundary expanders.

**Definition 2.12** (Coboundary expander). *A  $d$ -complex  $X$  is said to be  $\epsilon$ -coboundary expander,  $\epsilon > 0$ , if  $\text{Exp}_b^k(X) \geq \epsilon$  for any  $0 \leq k < d$ .*

This definition was first originate in the work of [LM], in connection to vanishing of (co)homological. Recall that for any  $0 \leq k \leq d$ , each coboundary is a cocycle, i.e.  $B^k(X; \mathbb{F}_2) \subset Z^k(X; \mathbb{F}_2)$ , and the  $k$ -th cohomology of  $X$  (in  $\mathbb{F}_2$ -coefficients) is the quotient space  $H^k(X; \mathbb{F}_2) = Z^k(X; \mathbb{F}_2)/B^k(X; \mathbb{F}_2)$ . The following simple equivalence holds:

$$\text{Exp}_b^k(X) > 0 \quad \Leftrightarrow \quad Z^k(X; \mathbb{F}_2) = B^k(X; \mathbb{F}_2) \quad \Leftrightarrow \quad H^k(X; \mathbb{F}_2) = 0.$$

Furthermore, if the  $k$ -th cohomology is trivial, then any cocycle is a coboundary, hence, in the case of vanishing of cohomology, the coboundary and the cocycle expansion parameters, are the same thing:

$$H^k(X; \mathbb{F}_2) = 0 \quad \Rightarrow \quad \text{Exp}_b^k(X) = \text{Exp}_z^k(X).$$

This imply the following equivalent condition for coboundary expansion, in terms of the cocycle expansion parameter and cohomology.

**Remark 2.13.** A  $d$ -complex  $X$  is an  $\epsilon$ -coboundary expander if and only if  $\text{Exp}_z^k(X) \geq \epsilon$  and  $Z^k = B^k$  for any  $0 \leq k < d$ .

As noted by Gromov (see also [DKW]), this notion of vanishing of cohomology is too strong for some application, since the existence of a cocycle which is not a coboundary, is acceptable just as long as it is not too small. This is where the definition of cosystoles come into play.

**Definition 2.14** (Systoles). Let  $X$  be a  $d$ -complex and  $0 \leq k \leq d$ . Define the  $k$ -cosystole of  $X$  to be the minimal size of a  $k$ -cocycle which is not a  $k$ -coboundary, i.e.

$$\text{Syst}^k(X) = \min_{z \in Z^k \setminus B^k} \|z\|. \quad (2.10)$$

We are now in a position to give the second definition of high dimensional expanders, the cosystolic expanders.

**Definition 2.15** (Cosystolic expander). A  $d$ -complex  $X$  is said to be  $(\epsilon, \mu)$ -cosystolic expander,  $\epsilon, \mu > 0$ , if  $\text{Exp}_z^k(X) \geq \epsilon$  and  $\text{Syst}^k(X) \geq \mu$  for any  $0 \leq k < d$ .

Note that a necessary condition for a complex to be an expander (both coboundary and cosystolic), is that all of its links are coboundary expanders.

**Definition 2.16** (Link coboundary expander). Let  $X$  be a simplicial complex. Call  $X$  a  $\beta$ -link coboundary expander if for any  $\emptyset \neq \sigma \in X$ , the link  $X_\sigma$  is a  $\beta$ -coboundary expander.

Let us now present an entirely different approach to high-dimensional expansion: Recall that an expander graph behave pseudo-randomly, a property captured by the mixing Lemma. Now, given a complex, consider the 1-dimensional skeletons (i.e. the underlying graphs) of the complex and its links, and say the complex is a skeleton-expander if all these graphs satisfy a good mixing property as follows:

**Definition 2.17** (Skeleton expander). A  $d$ -complex  $X$  is said to be  $\alpha$ -skeleton expander,  $1 > \alpha > 0$ , if for any  $\sigma \in X$  (including  $\sigma = \emptyset$ ), the 1-skeleton of the link,  $G = X_\sigma^{(1)}$ , satisfy the following mixing behaviour:

$$\forall A, B \subset X_\sigma(0), \quad \|E(A, B)\|_\sigma \leq 4 \cdot (\|A\|_\sigma \cdot \|B\|_\sigma + \alpha \cdot \sqrt{\|A\|_\sigma \cdot \|B\|_\sigma}) \quad (2.11)$$

where  $E(A, B) \subset X_\sigma(1)$  are the edges in  $X_\sigma$  with vertices from both  $A$  and  $B$ .

### 3 Skeletons Mixing Lemma

The purpose of this section is to prove a (half of a) mixing lemma for the skeleton of certain complexes, i.e. giving a spectral criterion for a skeleton-expander. Essentially what we prove here is one-side of a mixing lemma for graphs which are partite-regular.

Before proving such a mixing Lemma, it will be convenient to consider only complexes which are regular in the following sense.

**Definition 3.1** (Regular complex). A  $d$ -complex  $X$ , is said to be regular, if there exists a partition  $X(0) = \bigsqcup_{i=0}^d V_i$ , such that  $X(d) \subset \prod_{i=0}^d V_i$ , and furthermore for any  $I \subset J \subset [d]$ , there exists  $k_I^J \in \mathbb{N}$ , such that each  $p \in X \cap \prod_{i \in I} V_i$ , is contained in exactly  $k_I^J$  faces from  $X \cap \prod_{j \in J} V_j$ .

**Remark 3.2.** Note that if  $X$  is a regular complex, then so does all of its links.

Next, after defining what is a regular complex, we would like to specify which eigenvalues do we consider in its skeleton.

**Definition 3.3** (Non-trivial eigenvalue). *Let  $X$  be a regular  $d$ -complex. For any  $0 \leq i < j \leq d$ , define the  $(i, j)$ -type induced bipartite graph to be  $X_{(i,j)} = (V_i \sqcup V_j, X(1) \cap V_i \times V_j)$ . Denote by  $\lambda(X_{(i,j)})$  its normalized second largest eigenvalue. Define the normalized largest non-trivial eigenvalue of (the 1-skeleton of)  $X$  to be*

$$\lambda(X) := \max_{i \neq j} \lambda(X_{(i,j)}).$$

Considering the above notion of "second eigenvalue", we are now able to prove the following skeleton mixing Lemma.

**Proposition 3.4** (Skeleton Mixing Lemma). *Let  $X$  be a regular complex, and let  $\lambda(X)$  be its normalized largest non-trivial eigenvalue. Then,*

$$\forall A, B \subset X(0), \quad \|E(A, B)\| \leq 2\left(\frac{d+1}{d}\right) \cdot (\|A\| \cdot \|B\| + \lambda(X) \cdot \sqrt{\|A\| \cdot \|B\|}) \quad (3.1)$$

where  $E(A, B) \subset X(1)$  are the edges in  $X$  with vertices from both  $A$  and  $B$ .

Note that in the 1-dimensional case, a regular complex is the same as a bipartite biregular graph, and such a Mixing Lemma is already known.

**Lemma 3.5.** [EGL, Corollary 3.4] *Let  $G = (V_1 \sqcup V_2, E)$  be a bipartite biregular graph, and let  $\lambda(G)$  be its normalized second largest eigenvalue. Then,*

$$\forall A \subset V_1, B \subset V_2, \quad \left| \frac{|E(A, B)|}{|E(X)|} - \frac{|A|}{|V_1|} \frac{|B|}{|V_2|} \right| \leq \lambda(G) \cdot \sqrt{\frac{|A|}{|V_1|} \frac{|B|}{|V_2|}} \quad (3.2)$$

This bipartite mixing lemma will imply the general skeleton mixing lemma.

*Proof of Proposition 3.4.* First note that since  $X$  is a regular complex, and let  $X(0) = \bigsqcup_{i=0}^d V_i$  be the partition, then for any  $I \subset [d]$  and  $A \subset X \cap \prod_{i \in I} V_i$ ,

$$\binom{d+1}{|I|} \cdot \|A\| = \sum_{\sigma \in A} \frac{|\{F \in X(d) \mid \sigma \in F\}|}{|X(d)|} = \frac{|A| \cdot k_I^{[d]}}{|X \cap \prod_{i \in I} V_i| \cdot k_I^{[d]}} = \frac{|A|}{|X \cap \prod_{i \in I} V_i|}$$

In particular, for any  $i, j \in [d]$  and any  $A \subset V_i, B \subset V_j$ ,

$$\|A\| = \frac{1}{\binom{d+1}{1}} \frac{|A|}{|V_i|}, \quad \|B\| = \frac{1}{\binom{d+1}{1}} \frac{|B|}{|V_j|}, \quad \text{and} \quad \|E(A, B)\| = \frac{1}{\binom{d+1}{2}} \frac{|E(A, B)|}{|X(1) \cap V_i \times V_j|}$$

So restating Lemma 3.5 in terms of the norm, we get for any  $A \subset V_i, B \subset V_j$ ,

$$\begin{aligned} \|E(A, B)\| &\leq \frac{\binom{d+1}{1}^2}{\binom{d+1}{2}} \|A\| \cdot \|B\| + \lambda(X_{(i,j)}) \frac{\binom{d+1}{1}}{\binom{d+1}{2}} \sqrt{\|A\| \cdot \|B\|} \\ &\leq 2\left(\frac{d+1}{d}\right) \cdot (\|A\| \cdot \|B\| + \frac{\lambda(X)}{d+1} \cdot \sqrt{\|A\| \cdot \|B\|}) \end{aligned} \quad (3.3)$$

Now, let  $A, B \subset X(0)$ , and denote  $A_i = A \cap V_i$  and  $B_i = B \cap V_i$  for any  $0 \leq i \leq d$ . Since  $X$  is partite,  $E(A, B) = \bigsqcup_{i \neq j} E(A_i, B_j)$ , hence

$$\|E(A, B)\| = \sum_{i \neq j} \|E(A_i, B_j)\| \leq 2\left(\frac{d+1}{d}\right) \cdot \sum_{i \neq j} (\|A_i\| \cdot \|B_j\| + \frac{\lambda(X)}{d+1} \sqrt{\|A_i\| \cdot \|B_j\|})$$



Similarly, since  $A = \bigsqcup_i A_i$  and  $B = \bigsqcup_j B_j$ , then

$$\sum_{i \neq j} \|A_i\| \cdot \|B_j\| \leq \left( \sum_i \|A_i\| \right) \cdot \left( \sum_j \|B_j\| \right) = \|A\| \cdot \|B\| \quad (3.4)$$

Next, note that for any  $N$  non-negative numbers  $x_1, \dots, x_N \in \mathbb{R}_{\geq 0}$ , one has

$$\left( \sum_{i=1}^N x_i \right)^2 \leq N \cdot \max_{1 \leq i \leq N} (x_i^2) \leq N \cdot \sum_{i=1}^N x_i^2 \quad (3.5)$$

Applying this for  $N = (d+1)^2$  and  $x_i = \sqrt{\|A_i\| \cdot \|B_j\|}$ , we get

$$\sum_{i,j} \sqrt{\|A_i\| \cdot \|B_j\|} \leq \sqrt{t^2 \cdot \sum_{i,j} \|A_i\| \cdot \|B_j\|} \leq (d+1) \cdot \|A\| \cdot \|B\| \quad (3.6)$$

which finishes the proof.  $\square$

In particular, since  $2(\frac{d+1}{d}) \leq 4$  for any  $d \in \mathbb{N}$ , we get the following.

**Corollary 3.6.** *Let  $X$  be a regular  $d$ -complex and let  $\alpha = \max_{\sigma \in X} \lambda(X_\sigma)$ . Then  $X$  is an  $\alpha$ -skeleton expander.*

## 4 Main Theorems

The object of this section is to prove the following expansion criterion.

**Theorem 4.1** (Expansion criterion). *Let  $X$  be a  $d$ -complex which satisfy:*

- *$X$  is  $Q$ -bounded degree, i.e.  $|X_v| \leq Q$  for any  $v \in X(0)$ .*
- *$X$  is a  $\beta$ -link coboundary expander.*
- *$X$  is an  $\alpha$ -skeleton expander.*

*For any  $0 \leq k \leq d$ , there exists  $\epsilon = \epsilon(k, \beta, Q)$ ,  $\mu = \mu(k, \beta) > 0$  and  $\bar{\alpha} = \bar{\alpha}(k, \beta) > 0$  (for exact value see remark 4.4), such that, if  $\alpha \leq \bar{\alpha}$ , then*

$$\text{Exp}_z^k(X) \geq \epsilon \quad \text{and} \quad \text{Syst}^k(X) \geq \mu,$$

*i.e. the  $(d-1)$ -skeleton of  $X$  is an  $(\epsilon, \mu)$ -cosystolic expander.*

**Remark 4.2.** *Note that in Theorem 4.1, the requirement that all the links are coboundary expanders, cannot be weakened to the requirement that the links are merely cosystolic expanders, since any cocycle which is not a coboundary sitting in one of the links can be lifted to a cocycle which is not a coboundary of the entire complex, and it has very small size since it lives entirely in a single link, hence there is no hope for the general complex to have a cosystolic bound, in particular, the complex is not a cosystolic expander.*

In order to prove Theorem 4.1 we follow [KKL] strategy, who noticed that the following isoperimetric inequality for small cochains imply cosystolic expansion.

**Theorem 4.3** (Isoperimetric Inequality). *Let  $X$  be a  $d$ -complex which satisfy:*

- $X$  is a  $\beta$ -link coboundary expander.
- $X$  is an  $\alpha$ -skeleton expander.

For any  $0 \leq k \leq d$ , there exists  $\bar{\epsilon} = \bar{\epsilon}(k, \beta)$ ,  $\bar{\mu} = \bar{\mu}(k, \beta) > 0$  and  $\bar{\alpha} = \bar{\alpha}(k, \beta) > 0$  (for exact value see remark 4.4), such that, if  $\alpha \leq \bar{\alpha}$ , then

$$A \in C^k \text{ is locally minimal and } \|A\| \leq \bar{\mu} \implies \|\delta(A)\| \geq \bar{\epsilon} \cdot \|A\|$$

Note that the isoperimetric inequality in Theorem 4.3, unlike Theorem 4.1, does not require any bounded degree condition.

**Remark 4.4.** The constants in Theorem 4.3, are

$$\begin{aligned} \bar{\epsilon} &:= \frac{1}{3} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k, \\ \bar{\mu} &:= \left( \frac{1}{3} \cdot \frac{1}{(k+2)2^{k+3}} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k \right)^{2^{k+1}}, \\ \bar{\alpha} &:= \left( \frac{1}{3} \cdot \frac{1}{(k+2)2^{k+3}} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k \right)^{2^{k+1}+1}. \end{aligned}$$

The constants in Theorem 4.1, are

$$\begin{aligned} \mu &:= \left( \frac{1}{3} \cdot \frac{1}{(k+2)2^{k+3}} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k \right)^{2^{k+1}}, \quad \epsilon := \min\left\{\frac{1}{Q}, \mu\right\}, \\ \bar{\alpha} &:= \left( \frac{1}{3} \cdot \frac{1}{(k+2)2^{k+3}} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k \right)^{2^{k+1}+1}. \end{aligned}$$

## 4.1 Fat Faces

In order to prove the isoperimetric inequality (Theorem 4.3), we first construct a "fat-machinery" which allow us to move calculations from higher dimensions to lower dimensions, inside the complex.

We begin by defining the notion of fat faces. Essentially, for a fixed cochain, a face in the complex is fat if it is contained in many elements of the cochain.

**Definition 4.5** (Fat faces). Fix a cochain  $A \in C^k$  and a constant  $1 > \eta > 0$ . Define inductively the  $i$ -cochain of fat faces, w.r.t.  $A$  and  $\eta$ ,  $i = 0, \dots, k$ , by

$$S^k(A) = A \quad \text{and} \quad S^{i-1}(A) = \{\sigma \in X(i-1) \mid \|(S^i(A))_\sigma\|_\sigma \geq \eta^{2^{k-i}}\}$$

Call the elements of  $S^i(A)$ ,  $-1 \leq i \leq k$ , fat faces (w.r.t.  $A$  and the fatness constant  $\eta$ ).

Very roughly, one can say that our strategy for proving Theorem 4.3, is as follows: "move information from  $A = S^k(A) \subset X(k)$  to  $S^{-1}(A) \subset X(-1)$ ".

The first result in this direction is the following Lemma, which says that sizes of the cochains of fat faces is bounded by the size of the original cochain (up to some constant).

**Lemma 4.6** (Fat size Lemma). *Let  $X$  be a  $d$ -complex,  $-1 \leq i \leq k \leq d$ ,  $A \in C^k$  and  $1 > \eta > 0$ . Then,*

$$\|S^i(A)\| \leq \eta^{-2^{k-i}} \cdot \|A\| \quad (4.1)$$

*Proof.* For any  $-1 \leq j \leq k-1$  and any fat  $j$ -face  $\sigma \in S^j(A)$ , by applying Lemma 2.6 on the cochain  $(S^{j+1}(A))_\sigma \in C_\sigma^0$ , we get

$$w(\sigma) = \frac{\|(\widetilde{S^{j+1}(A)})_\sigma\|}{(j+2) \cdot \|(S^{j+1}(A))_\sigma\|_\sigma} \leq ((j+2) \cdot \eta^{2^{k-j-1}})^{-1} \cdot \|(\widetilde{S^{j+1}(A)})_\sigma\| \quad (4.2)$$

Hence, combining this with Lemma 2.7,

$$\|S^j(A)\| = \sum_{\sigma \in S^j(A)} w(\sigma) \leq \frac{\eta^{-2^{k-j}}}{j+2} \cdot \sum_{\sigma \in S^j(A)} \|(\widetilde{S^{j+1}(A)})_\sigma\| = \eta^{-2^{k-j-1}} \cdot \|(S^{j+1}(A))\| \quad (4.3)$$

Hence, by iterating on equation (4.3) for  $j = i, \dots, k-1$ , we get,

$$\begin{aligned} \|S^i(A)\| &\leq \eta^{-2^{k-i-1}} \cdot \|(S^{i+1}(A))\| \leq \eta^{-(2^{k-i-1} + 2^{k-i-2})} \cdot \|(S^{i+2}(A))\| \leq \dots \\ &\dots \leq \eta^{-\sum_{j=i}^{k-1} 2^{k-j}} \|S^k(A)\| = \eta^{-2^{k-i} + 1} \cdot \|A\| \leq \eta^{-2^{k-i}} \cdot \|A\| \end{aligned} \quad (4.4)$$

which finishes the proof.  $\square$

From Lemma 4.6, we get the following consequence, which says that for a small cochain the unique  $(-1)$ -face is a non-fat face. (This simple fact will serve as the finishing argument in the proof of Theorem 4.3).

**Corollary 4.7.** *Let  $X$  be a  $d$ -complex,  $k \leq d$ ,  $A \in C^k$  and  $1 > \eta > 0$ . If  $\|A\| < \eta^{-2^{k+1}}$  then the unique  $(-1)$ -face, the empty set, is not fat.*

*Proof.* Assume in contradiction that  $\emptyset \in X(-1)$  is fat. Note that the empty set has the following interesting property: "A local view by  $\emptyset$  is everything", i.e. for any  $Y \subset X$  then  $Y_\emptyset = Y$ . So, assuming  $\emptyset$  is fat, we get by definition that,

$$\|S^0(A)\| = \|(S^0(A))_\emptyset\|_\emptyset \geq \eta^{2^k} \quad (4.5)$$

However, from Lemma 4.6 and the assumption on the size of  $A$ , we have

$$\|S^0(A)\| \leq \eta^{-2^k} \cdot \|A\| < \eta^{2^k} \quad (4.6)$$

which leads to a contradiction, hence  $\emptyset \in X(-1)$  is not fat.  $\square$

Next, we define the cochain of degenerate faces, which intuitively one should think of as the error-term when one is trying to move from higher dimension to lower dimension.

**Definition 4.8** (Degenerate faces). *Fix a cochain  $A \in C^k$  and  $1 > \eta > 0$ .*

*A dead-end is a pair of two equal sized fat faces,  $(\sigma, \sigma')$ , whose intersection is a codimension-1 non-fat face, i.e.*

$$|\sigma| = |\sigma'| = |\sigma \cap \sigma'| + 1, \quad \sigma, \sigma' \in S^*(A), \quad \sigma \cap \sigma' \notin S^{*-1}(A)$$

*A face  $p \in X$  is said to be degenerate if it contains a dead-end in it, and define  $\Upsilon(A) \in C^{k+1}$  to be the cochain of all  $(k+1)$ -faces which are degenerate.*

The following Proposition gives an effective bound on the cochain of degenerate faces in terms of the skeleton expansion and the fatness constant.

**Proposition 4.9** (Fat mixing Lemma). *Let  $X$  be a  $d$ -complex which is a  $\alpha$ -skeleton expander,  $\alpha > 0$ . For any  $k$ -cochain  $A$  and fatness constant  $1 > \eta > 0$ , then*

$$\|\Upsilon(A)\| \leq (k+2) \cdot 2^{k+3} \cdot (\eta + \alpha \cdot \eta^{-2^k}) \cdot \|A\| \quad (4.7)$$

*Proof.* For any  $t \leq d$  and any  $t$ -cochain  $Y$ , denote by  $E(Y, Y)$  the  $(t+1)$ -cochain of  $(t+1)$ -faces which contains at least two different  $t$ -faces from  $Y$ , and let  $\Gamma^r(Y) \in C^r$  be as in Lemma 2.3. Then, by the definition of the fat-degenerate faces, we get

$$\Upsilon(A) \subset \bigcup_{i=-1}^{k-1} \bigcup_{\sigma \in X(i) \setminus S^i(A)} \Gamma^{k+1}(E((S^{i+1}(A))_\sigma, (S^{i+1}(A))_\sigma)) \quad (4.8)$$

So, by Lemma 2.3 and the triangle inequality, we get

$$\|\Upsilon(A)\| \leq \sum_{i=-1}^{k-1} \sum_{\sigma \in X(i) \setminus S^i(A)} \binom{k+2}{i+2} \cdot \|E((S^{i+1}(A))_\sigma, (S^{i+1}(A))_\sigma)\| \quad (4.9)$$

From the skeleton expansion, we get for any  $\sigma \in X(i)$ ,

$$\|E((S^{i+1}(A))_\sigma, (S^{i+1}(A))_\sigma)\|_\sigma \leq 4 \cdot \|(S^{i+1}(A))_\sigma\|_\sigma \cdot (\|(S^{i+1}(A))_\sigma\|_\sigma + \alpha) \quad (4.10)$$

Now, by the Lemma 2.6, we can multiply both sides by  $\binom{k}{|\sigma|} \cdot w(\sigma)$ , and get

$$\|E((S^{i+1}(A))_\sigma, (S^{i+1}(A))_\sigma)\| \leq 4 \cdot \|(S^{i+1}(A))_\sigma\| \cdot (\|(S^{i+1}(A))_\sigma\|_\sigma + \alpha) \quad (4.11)$$

Next, if  $\sigma \in X(i) \setminus S^i(A)$ , then by the definition of fat faces,

$$\|E((S^{i+1}(A))_\sigma, (S^{i+1}(A))_\sigma)\| \leq 4 \cdot \|(S^{i+1}(A))_\sigma\| \cdot (\eta^{2^{k-i}} + \alpha) \quad (4.12)$$

Summing this over all non-fat  $i$ -faces,

$$\begin{aligned} \sum_{\sigma \in X(i) \setminus S^i(A)} \|E((S^{i+1}(A))_\sigma, (S^{i+1}(A))_\sigma)\| &\leq 4(\eta^{2^{k-i}} + \alpha) \cdot \sum_{\sigma \in X(i) \setminus S^i(A)} \|(S^{i+1}(A))_\sigma\| \\ &\leq 4(\eta^{2^{k-i}} + \alpha) \cdot \sum_{\sigma \in X(i)} \|(S^{i+1}(A))_\sigma\| = 4(\eta^{2^{k-i}} + \alpha) \cdot (i+2) \cdot \|S^{i+1}(A)\| \end{aligned} \quad (4.13)$$

where the last equality follows from Lemma 2.7. Applying Lemma 4.6, we get

$$\sum_{\sigma \in X(i) \setminus S^i(A)} \|E((S^{i+1}(A))_\sigma, (S^{i+1}(A))_\sigma)\| \leq 4(i+2) \cdot (\eta^{2^{k-(i+1)}} + \alpha \cdot \eta^{-2^{k-(i+1)}}) \cdot \|A\| \quad (4.14)$$

Combining equations (4.9) and (4.14) together, we get

$$\begin{aligned} \|\Upsilon(A)\| &\leq \sum_{i=-1}^{k-1} \binom{k+2}{i+2} \cdot 4(i+2) \cdot (\eta^{2^{k-(i+1)}} + \alpha \cdot \eta^{-2^{k-(i+1)}}) \cdot \|A\| \\ &\leq \left( 4 \cdot \sum_{i=-1}^{k-1} \binom{k+2}{i+2} \cdot (i+2) \right) \cdot (\eta^{2^{k-k}} + \alpha \cdot \eta^{-2^{k-0}}) \cdot \|A\| \\ &\leq \left( 4(k+2) \cdot \sum_{i=-1}^{k-1} \binom{k+1}{i+1} \right) \cdot (\eta + \alpha \cdot \eta^{-2^k}) \cdot \|A\| \leq (k+2) \cdot 2^{k+3} \cdot (\eta + \alpha \cdot \eta^{-2^k}) \cdot \|A\| \end{aligned} \quad (4.15)$$

which finishes the proof.  $\square$

Next, we define the notion of a fat ladder.

**Definition 4.10** (Fat ladders). *Fix a cochain  $A \in C^k$  and a constant  $1 > \eta > 0$ . For any fat  $i$ -face,  $\sigma \in S^i(A)$ , define the  $k$ -cochain of fat-ladders siting on  $\sigma$ , to be*

$$L(A, \sigma) = \{t \in A \mid \exists \sigma = \sigma_i \subset \dots \subset \sigma_0 = t, \sigma_j(t) \in S^j(A), \forall j = 0, \dots, i\}$$

*Define the  $k$ -cochain of  $i$ -fat-ladders by  $L(A, i) = \cup_{\sigma \in S^i(A)} L(A, \sigma)$ .*

The following Lemma is the key idea behind Lemma 1.18 from the introduction. Figuratively speaking, the Lemma says that either we can "climb down" a fat-ladder step by step (where step means a fat face) from the highest level to the lowest level in the ladder, or we are in a fat-degenerate situation.

**Lemma 4.11** (Ladders Lemma). *Let  $X$  be a  $d$ -complex,  $-1 \leq i \leq k \leq d$ ,  $A \in C^k$  and  $1 > \eta > 0$ . Let  $p \in X(k+1)$ ,  $\sigma \in S^i(A)$ , and assume  $p$  contains  $t \in L(A, \sigma)$  and  $t' \in A$ . Then either  $t' \in L(A, t' \cap \sigma)$  or  $p \in \Upsilon(A)$ .*

*Proof.* By definition, there exists  $\sigma = \sigma_i \subsetneq \dots \subsetneq \sigma_0 = t$ , where all  $\sigma_j$  are fat. Define  $\sigma'_0 = t'$  and  $\sigma'_{j+1} = t' \cap \sigma_j = \sigma'_j \cap \sigma_j$  for any  $j = 0, \dots, i$ .

If all the  $\sigma'_j$  are fat, and since  $t' \cap \sigma = \sigma'_{i+1} \subset \dots \subset \sigma'_0 = t'$ , so by removing repetitions if needed ( $\sigma'_j = \sigma'_{j+1}$ ) we get that  $t' \in L(A, t' \cap \sigma)$ .

Otherwise, there is a non-fat  $\sigma'_j$ , and w.l.o.g. we may assume  $j$  is minimal, i.e.  $\sigma'_{j-1}$  is fat, and since that  $\sigma_{j-1} \not\subset t'$  (otherwise  $\sigma'_j = \sigma_{j-1}$ ), we get that  $|\sigma_{j-1}| = |\sigma'_{j-1}| = |\sigma_{j-1} \cap \sigma'_{j-1}| - 1$ , hence  $p \in \Upsilon(A)$ .  $\square$

## 4.2 Proof of the isoperimetric Inequality (Theorem 4.3)

After setting the "fat machinery", we are able to prove the following formal version of Lemma 1.19 from the introduction. Note that in Lemma 1.19 there is no mentioning of the cochain of fat-degenerate faces  $\Upsilon(A)$ , this is because Proposition 4.12 promise us that its contribution can be negligible (assuming good skeleton expansion).

**Proposition 4.12** (Seeping Lemma). *Let  $X$  be an  $\beta$ -link coboundary expander. Fix a locally minimal  $k$ -cochain  $A$  and  $1 > \eta > 0$ . Then for any  $0 \leq i \leq k$ ,*

$$\frac{\beta}{\binom{k+2}{i+1}} \cdot \|L(A, i)\| \leq \|\delta(A)\| + (k+2) \cdot \|L(A, i-1)\| + \|\Upsilon(A)\| \quad (4.16)$$

*Proof.* Let us evaluate the following expression,

$$R = \bigcup_{\sigma \in S^i(A)} \delta_\sigma(L(A, \sigma)) \subset X(k+1). \quad (4.17)$$

On the one hand: Since  $A$  is locally minimal, i.e.  $A_\sigma$  is minimal, and by Lemma 2.9,  $L(A, \sigma) \subset A_\sigma$  is also minimal. So by the assumption that the links  $X_\sigma$  are  $\beta$ -coboundary-expanders, we get

$$\beta \cdot \|L(A, i)\| \leq \sum_{\sigma \in S^i(A)} \beta \cdot \|L(A, \sigma)\| \leq \sum_{\sigma \in S^i(A)} \|\delta_\sigma(L(A, \sigma))\| \leq \binom{k+2}{i+1} \cdot \|R\| \quad (4.18)$$

On the other hand: For any  $\sigma \in S^i(A)$  and any  $p \in \delta_\sigma(L(A, \sigma))$ , one of the following three possibilities occur:

1. All the  $k$ -faces in  $p$  which belongs to  $A$  contain  $\sigma$ , and they are all from  $L(A, \sigma)$ . In which case  $p \in \delta(A)$ .
2. All the  $k$ -faces in  $p$  which belongs to  $A$  contain  $\sigma$ , but not all of them are from  $L(A, \sigma)$ . I.e. there is some  $t' \in A \setminus L(A, \sigma)$  such that  $\sigma \subset t' \subset p$ . Now, since  $p \in \delta_\sigma(L(A, \sigma))$ , there must be at least one  $t \in L(A, \sigma)$  such that  $t \subset p$ . So, by Lemma 4.11 we get that  $p \in \Upsilon(A)$ .
3. There is a  $k$ -face in  $p$  which belongs to  $A$  and does not contain  $\sigma$ . I.e. there is some  $t' \in A$  such that  $\sigma \not\subset t' \subset p$ . Like before, since  $p \in \delta_\sigma(L(A, \sigma))$ , there must be at least one  $t \in L(A, \sigma)$  such that  $t \subset p$ . And again by Lemma 4.11 we get that either  $p \in \Upsilon(A)$ , or  $t' \in L(A, t' \cap \sigma) \subset L(A, i-1)$ , i.e.  $p$  is a  $(k+1)$ -face which contains a  $k$ -face from  $L(A, i-1)$ , hence by definition 2.2  $p \in \Gamma^{k+1}(L(A, i-1))$ .

In conclusion we get that  $R \subset \delta(A) \cup \Gamma^{k+1}(L(A, i-1)) \cup \Upsilon(A)$ .

Combining both these estimates of  $R$ , together with Lemma 2.3, yields

$$\frac{\beta}{\binom{k+2}{i+1}} \cdot \|L(A, i)\| \leq \|R\| \leq \|\delta(A)\| + (k+2) \cdot \|L(A, i-1)\| + \|\Upsilon(A)\| \quad (4.19)$$

which finishes the proof.  $\square$

Finally, we are able to prove the isoperimetric inequality, where of course the two key ingredients in this proof are Propositions 4.12 and 4.9.

*Proof of Theorem 4.3.* Let  $1 > \eta > 0$  be the fatness constant which will be defined later, and define  $\bar{\mu} := \eta^{2^{k+1}}$ . Note that by definition  $L(A, k) = A$ , and by the assumption  $\|A_\emptyset\|_\emptyset = \|A\| \leq \bar{\mu} = \eta^{2^{k+1}}$  we get by Corollary 4.7 that the only  $(-1)$ -face, the empty set, is non fat (w.r.t.  $A$  and  $\eta$ ), and hence  $\|L(A, -1)\| = 0$ . Therefore, for any constant  $c \leq 1$ , we get,

$$c^k \cdot \|A\| = \sum_{i=0}^k c^{k-i} (c \cdot \|L(A, i)\| - \|L(A, i-1)\|) \leq \sum_{i=0}^k (c \cdot \|L(A, i)\| - \|L(A, i-1)\|) \quad (4.20)$$

Define  $c_0 := \frac{\beta}{(k+2) \cdot 2^{k+2}}$ , and note that  $(k+2) \cdot c_0 \leq \frac{\beta}{\binom{k+2}{i+1}}$ , for any  $0 \leq i \leq k$ . So, by applying Proposition 4.12 on equation (4.20), and the constant  $c_0$ , we get

$$c_0^k \cdot \|A\| \leq \frac{1}{k+2} \cdot \sum_{i=0}^k \left( \frac{\beta}{\binom{k+2}{i+1}} \cdot \|L(A, i)\| - (k+2) \cdot \|L(A, i-1)\| \right) \leq \|\delta(A)\| + \|V(A)\| \quad (4.21)$$

Combining this with Proposition 4.9,

$$\|\delta(A)\| \geq (c_0^k - (k+2)2^{k+3} \cdot (\eta + \alpha \cdot \eta^{-2^{k+1}})) \cdot \|A\| \quad (4.22)$$

Finally, by defining the constants,

$$\begin{aligned}\bar{\epsilon} &:= \frac{1}{3} \cdot c_0^k = \frac{1}{3} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k, \\ \eta &:= \frac{1}{(k+2)2^{k+3}} \cdot \bar{\epsilon} = \frac{1}{3} \cdot \frac{1}{(k+2)2^{k+3}} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k, \\ \bar{\mu} &:= \eta^{2^{k+1}} = \left( \frac{1}{3} \cdot \frac{1}{(k+2)2^{k+3}} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k \right)^{2^{k+1}} \\ \bar{\alpha} &:= \eta^{2^{k+1}+1} = \left( \frac{1}{3} \cdot \frac{1}{(k+2)2^{k+3}} \cdot \left( \frac{\beta}{(k+2) \cdot 2^{k+2}} \right)^k \right)^{2^{k+1}+1}.\end{aligned}$$

So, if the skeleton expansion parameter satisfy  $\alpha \leq \bar{\alpha}$ , then equation (4.22) reads  $\|\delta(A)\| \geq \bar{\epsilon} \cdot \|A\|$ , which finishes the proof.  $\square$

### 4.3 Proof of the Expansion criterion (Theorem 4.1)

The fact that an isoperimetric inequality for small cochains (Theorem 4.3), implies a cosystolic expansion (Theorem 4.1), was first shown in [KKL, § 4], but for the sake of being self-contained we add here their argument.

**Proposition 4.13.** [KKL, Proposition 2.5] *Let  $X$  be a  $d$ -complex, and define  $Q = \max_{v \in X(0)} |X_v|$  ("the degree of  $X$ "). Let  $0 \leq k \leq d$ .*

*Then for any  $A \in C^k$ , there exists  $\gamma \in C^{k-1}$ , which satisfies:*

*1)  $A + \delta(\gamma)$  is locally minimal, 2)  $\|A + \delta(\gamma)\| \leq \|A\|$ , and 3)  $\|\gamma\| \leq Q \cdot \|A\|$ .*

*Proof.* First note that  $N(A) := \binom{d+1}{k+1} \cdot |X(d)| \cdot \|A\|$  is a non-negative integer. We prove the claim by induction on  $N(A)$ . In the base case  $N(A) = 0$ , then  $A = \emptyset \in C^k$  is the empty  $k$ -cochain, and the claim holds for  $\gamma = \emptyset \in C^{k-1}$  the empty  $(k-1)$ -cochain. Assume the claim holds for all cochains  $A' \in C^k$  such that  $N(A') < N(A)$ , i.e. such that  $\|A'\| < \|A\|$ .

If  $A$  is locally minimal, then the claim holds for  $\gamma = \emptyset \in C^{k-1}$  the empty  $(k-1)$ -cochain. Otherwise, there exist  $\emptyset \neq \sigma \in X$ , and some  $c \in C_{\sigma}^{k-|\sigma|-1} \subset C^{k-1}$ , such that

$$\sum_{\sigma \subset \tau \in A + \delta(c)} w(\tau) = \|A_{\sigma} + \delta(c)\| < \|A_{\sigma}\| = \sum_{\sigma \subset \tau \in A} w(\tau) \quad (4.23)$$

Now, since for any  $\sigma \not\subset \tau$ ,  $\tau \in A + \delta(c)$  if and only if  $\sigma \not\subset \tau \in A$ , then

$$\|A + \delta(c)\| = \sum_{\sigma \subset \tau \in A + \delta(c)} w(\tau) + \sum_{\sigma \not\subset \tau \in A + \delta(c)} w(\tau) < \sum_{\sigma \subset \tau \in A} w(\tau) + \sum_{\sigma \not\subset \tau \in A} w(\tau) = \|A\|. \quad (4.24)$$

So  $N(A + \delta(c)) < N(A)$ , and by the induction assumption there exist  $\gamma' \in C^{k-1}$ , such that: 1)  $A + \delta(c + \gamma') = (A + \delta(c)) + \delta(\gamma')$  is a locally minimal cochain, 2)  $\|A + \delta(c + \gamma')\| \leq \|A + \delta(c)\| < \|A\|$ , and 3)  $\|\gamma'\| \leq Q \cdot \|A + \delta(c)\| \leq Q \cdot (\|A\| - 1)$ . Hence by taking  $\gamma = \gamma' + c$ , and noting that  $\|c\| \leq Q$  (since  $c \subset X_{\sigma}$ ), we get the claim for  $A$ , which finishes the proof.  $\square$

Now, using Proposition 4.13 and the isoperimetric inequality (Theorem 4.3), we are able to prove the cosystolic expansion criterion (Theorem 4.1).

*Proof of Theorem 4.1.* Define  $\epsilon = \min\{\bar{\mu}, \frac{1}{Q}\}$  and  $\mu = \bar{\mu}$ , where  $\bar{\mu}$  (and  $\bar{\epsilon}$ ) are the constants from Theorem 4.3 and  $Q = \max_{v \in X(0)} |X_v|$ .

We begin by proving the cocycle expansion. Let  $A \in C^k$ . First note that if  $\|\delta(A)\| \geq \bar{\mu}$ , and since  $\|A\| \leq 1$  for any cochain, then

$$\|\delta(A)\| \geq \bar{\mu} \geq \epsilon \geq \epsilon \cdot \|A\| \quad (4.25)$$

So let us assume  $\|\delta(A)\| \leq \mu$ , and let  $\gamma \in C^k$ . By as in Proposition 4.13. Then  $\delta(A + \gamma)$  is a locally minimal cochain and  $\|\delta(A + \gamma)\| \leq \|\delta(A)\| \leq \bar{\mu}$ , so by Theorem 4.3 and the fact that  $\delta \circ \delta = 0$  we get that

$$0 = \|\delta(\delta(A + \gamma))\| \geq \bar{\epsilon} \cdot \|\delta(A + \gamma)\| \Rightarrow \delta(A + \gamma) = 0. \quad (4.26)$$

So  $A + \gamma \in Z^k$ , and hence  $\gamma = A + (A + \gamma) \in \{A + z \mid z \in Z^k\}$ . Now, by Proposition 4.13,  $\|\gamma\| \leq Q \cdot \|\delta(A)\|$ , and we get

$$\|\delta(A)\| \geq \frac{1}{Q} \cdot \|\gamma\| \geq \epsilon \cdot \|\gamma\| = \epsilon \cdot \|A + (A + \gamma)\| \geq \epsilon \cdot \min_{z \in Z^k} \|A + z\| \quad (4.27)$$

which gives us the cocycle expansion  $\text{Exp}_z^k(X) \geq \epsilon$ .

Next we prove the cosystolic bound. Let  $A \in Z^k \setminus B^k$  (if no such cocycle exists there is nothing to prove). By Proposition 4.13, let  $A' = A + \delta(\gamma)$  be such that  $A'$  is locally minimal and  $\|A'\| \leq \|A\|$ . Note that since  $A \in Z^k \setminus B^k$  and  $\delta(\gamma) \in B^k$  then  $A' \in Z^k \setminus B^k$  also. If  $\|A'\| \leq \mu = \bar{\mu}$ , then by Theorem 4.3 and the fact that  $A'$  is a cocycle, we get

$$0 = \|\delta(A')\| \geq \bar{\epsilon} \cdot \|A'\| \Rightarrow A' = 0 \quad (4.28)$$

which is a contradiction since  $0 \in B^k$  and  $A'$  is not. So  $\|A\| \geq \|A'\| \geq \mu$ , which gives us the cosystolic bound  $\text{Syst}^k(X) \geq \mu$ .  $\square$

## 5 Spherical Buildings

The object of this section is to introduce the notion of spherical buildings, and to show that they are good skeleton expanders.

### 5.1 Definition of spherical Buildings

Here we give a definition of spherical buildings, and list some of their properties which we shall use. For more on buildings we refer to [AB].

Before defining a building, let us first define the notion of a chamber complex.

**Definition 5.1** (Chamber complex). *A  $d$ -dimensional simplicial complex,  $X$ , is said to be a chamber complex, if it is pure (i.e. all maximal faces are  $d$ -dimensional), and for any two maximal faces  $C, C' \in X(d)$ , there is a sequence of  $d$ -faces,  $C = C_1, \dots, C_n = C'$ , such that for each  $i = 1, \dots, n-1$ , the intersection  $C_i \cap C_{i+1}$  is a  $(d-1)$ -face.*

*For a  $d$ -dimensional chamber complexes, it is custom to call a  $d$ -face a chamber, to call a  $(d-1)$ -face a panel, and the above sequence  $C = C_1, \dots, C_n = C'$  is called a gallery from  $C$  to  $C'$ .*

*A chamber complex,  $X$ , is said to be thin if each panel is contained in exactly 2 chambers, and it is said to be  $q$ -thick,  $q > 1$ , if each panel is contained in exactly  $q+1$  chambers. By a thick building we mean a  $q$ -thick for some  $q > 1$ .*



Then, one way to define a building is as follows (for the equivalence for the more common definition, see [AB, Theorem 4.131]).

**Definition 5.2** (Building). *A building is a thick chamber complex together with a family of subcomplexes, called apartments, which satisfy the following axioms:*

- *Each apartment is a thin chamber complex.*
- *Any two faces in the complex are contained in a common apartment.*
- *Any two apartments have an isomorphism which fixes their intersection.*

*A building is said to be spherical if it is finite.*

Let us note that if  $B$  is a  $d$ -dimensional building (i.e. a  $d$ -dimensional chamber complex which satisfy the axioms of the building), then each apartment of  $B$  is also  $d$ -dimensional.

**Remark 5.3.** *Throughout this paper we only concerns ourselves with buildings which are simplicial complexes. However, it should be mentioned that buildings need not be simplicial complexes, they can also be poly-simplicial complexes (just by allowing chamber complexes to be such).*

Let us present an example of a spherical building.

**Example 5.4.** [AB, § 4.3] *Let  $q$  be a prime power,  $d \in \mathbb{N}$  and denote  $V = \mathbb{F}_q^d$ . Consider the simplicial complex,  $\mathbb{P}(V)$ , whose vertices are the proper (i.e. not  $\{0\}$  or  $V$ ) subspaces of  $V$ , and his faces are the flags of subspaces in  $V$ , i.e.  $\{0\} < W_1 < \dots < W_t < V$ . Then  $\mathbb{P}(V)$  is a spherical building. Moreover, the group  $PGL_d(\mathbb{F}_q)$  acts on  $\mathbb{P}(V)$  in a strongly transitive way (see below).*

Next, we wish to list some basic properties of spherical buildings, for the complete proofs we refer to [AB].

**Lemma 5.5.** *Each apartment in a  $d$ -dimensional building is of size at most  $C(d) := \max\{2^d \cdot (d+1)!, 192 \cdot 11!\}$ .*

*Proof.* By [AB, Theorem 4.131] each apartment in a spherical building is a spherical Coxeter complex, and by [AB, § 1.3, 1.5.6] the spherical Coxeter complexes were completely classified, and  $C(d)$  is taken to be the maximal size of all such possible complexes.  $\square$

**Definition 5.6** (Type function). *A  $d$ -complex,  $X$ , is said to admit a  $(d+1)$ -type function on the vertices if there is a function  $\tau_X : X(0) \rightarrow [d] := \{0, 1, \dots, d\}$ , such that, setting  $V_i := \tau_X^{-1}(\{i\})$  for  $i = 0, \dots, d$ , then  $X(0) = \bigsqcup_{i=0}^d V_i$  and  $X(d) \subset \prod_{i=0}^d V_i$ .*

**Lemma 5.7.** [AB, Proposition 4.6] *Let  $B$  be a  $d$ -dimensional building. Then  $B$  admits a  $(d+1)$ -type function on its vertices.*

**Lemma 5.8.** [AB, Proposition 4.9] *Let  $B$  be a building and let  $\sigma \in B$  be any face in it. Then the link,  $B_\sigma$ , is also a building.*

**Lemma 5.9.** [AB, Proposition 4.40] *Let  $B$  be a building and let  $A$  be an apartment in it. Let  $C, C' \in A$  be two chambers which sits in  $A$ . If  $C = C_0, \dots, C_n = C'$  is a gallery from  $C$  to  $C'$  in the building, and this gallery is of minimal length among all possible galleries from  $C$  to  $C'$ , then the gallery sits completely inside the apartment  $A$ , i.e.  $C_0, \dots, C_n \in A$ .*

**Lemma 5.10.** [AB, Proposition 5.122 (2)] Let  $B$  be a spherical building and let  $C$  be a chamber in  $B$ . For any apartment containing  $C$ ,  $A$ , there is a unique chamber in  $A$ , denoted  $C_A^{op}$ , which is of maximal gallery distance from  $C$ .

Next we define a notion of a building which possesses many symmetries.

**Definition 5.11** (Strongly transitive action). A building is said to possess a strongly transitive action, if there exist a group of automorphisms on the building  $G \leq \text{Aut}(B)$ , such that:

- $G$  preserves the  $(d+1)$ -type function of the building as defined in Lemma 5.7.
- For any two pairs,  $(C_1, A_1)$  and  $(C_2, A_2)$ , of a chamber,  $C_i$ , and an apartment containing the chamber,  $A_i$ ,  $i = 1, 2$ , there exists  $g \in G$ , such that  $g(C_1) = C_2$  and  $g(A_1) = A_2$ .

**Remark 5.12.** Due to a remarkable result of Tits, all spherical building of dimension greater than two possess a strongly transitive action.

**Lemma 5.13.** Let  $B$  be a  $d$ -dimensional building and  $G$  a group that acts strongly transitively on it. Then  $B$  is a regular complex (see §3).

*Proof.* By Lemma 5.7, there exists a type-function  $\tau_B : B(0) \rightarrow [d]$ , i.e. if  $V_i := \tau_B^{-1}(\{i\}) \subset B(0)$ ,  $i = 0, \dots, d$ , then  $B(0) = \bigsqcup_{i=0}^d V_i$  and  $B(d) \subset \prod_{i=0}^d V_i$ . Now, let  $I \subset J \subset [d]$  be two type sets, and let  $\sigma, \sigma' \in B \cap \prod_{i \in I} V_i$  be two  $I$ -type faces. Choosing two chambers,  $C$  and  $C'$ , which contains  $\sigma$  and  $\sigma'$  respectively, and by the second property of the strong transitivity there exists  $g \in G$  such that  $g(C) = C'$ . Also by the first property of the strong transitivity,  $g$  preserves the types of  $\tau_B$ , hence  $g(\sigma) = \sigma'$ . Since  $g$  is an automorphism, the  $J$ -type faces containing  $\sigma$  are mapped bijectively to the  $J$ -type faces containing  $\sigma'$ , in particular they are of the same cardinality, proving that the building is regular.  $\square$

Throughout this section we shall make use of the following notion from group theory.

**Definition 5.14** (Stabilizer). Let  $X$  be any simplicial complex and  $G \leq \text{Aut}(X)$  a group of automorphisms on  $X$ . For any  $\sigma \in X$ , define the stabilizer of  $\sigma$  in  $G$  to be the following subgroup of  $G$ :

$$G_\sigma = \text{stab}_G(\sigma) = \{g \in G \mid g(\sigma) = \sigma\}.$$

**Lemma 5.15.** Let  $B$  be a building and  $G$  a group that acts strongly transitively on it. Then for any face,  $\sigma$ , and any apartment containing it,  $A$ , every  $G_\sigma$ -orbit in  $B$  passes through the apartment  $A$ , i.e. for any  $\tau \in B$  there exists  $g \in G_\sigma$  such that  $g(\tau) \in A$ .

*Proof.* Let  $C \in A$  be a maximal face containing  $\sigma$ , and let  $C' \in B$  be a maximal face containing  $\tau$ . By the second axiom of the building, there exists an apartment  $A'$  which contains both  $C$  and  $C'$ . By the strong-transitivity, there exists  $g \in G$  such that  $g(C) = C$  and  $g(A') = A$ . On the hand  $g$  preserves the type function, hence  $g(\sigma) = \sigma$ , i.e.  $g \in \text{stab}_G(\sigma)$ . On the other hand  $g(A') = A$ , in particular  $g.C' \in A$ , as needed.  $\square$

**Lemma 5.16.** Let  $B$  be a spherical building which possesses a strongly transitive action, and let  $\sigma \in B$ . Then the link  $B_\sigma$  also possesses a strongly transitive action.

*Proof.* Let  $G \leq \text{Aut}(B)$  be the group that acts strongly transitively on  $B$ . Define  $G_\sigma = \text{stab}_G(\sigma)$  the stabilizer of  $\sigma$  in  $G$ , then  $G_\sigma$  admits an action on the link, i.e.  $G_\sigma \leq \text{Aut}(B_\sigma)$ . The action of  $G_\sigma$  is type-preserving since the action of  $G$  is. Any pair of a chamber and an apartment containing it inside  $B_\sigma$ , ( $C \in A$ ), can be lifted uniquely to a pair of a chamber and an apartment

containing it inside  $B$ ,  $(\tilde{C} \in \tilde{A})$ . Let  $(C \in A)$  and  $(C' \in A')$ , be two pairs, where each pair contains a chamber and an apartment containing that chamber, inside  $B_\sigma$ . Lifting them to such pairs in  $B$ ,  $(\tilde{C} \in \tilde{A})$  and  $(\tilde{C}' \in \tilde{A}')$ , then there is  $g$  in  $G$  such that  $g.(\tilde{C} \in \tilde{A}) = (\tilde{C}' \in \tilde{A}')$ , and since both pairs contains  $\sigma$ , then  $g \in G_\sigma$ , and hence  $g(C) = C'$  and  $g(A) = A'$ , which finishes the proof.  $\square$

## 5.2 Expansion of Spherical Buildings

It was first observed by Gromov [G] that spherical buildings are coboundary expanders (see [LMM] for a proof).

**Theorem 5.17.** *For any  $d \in \mathbb{N}$ , there exist  $\beta = \beta(d) > 0$ , such that any  $d$ -dimensional spherical building is a  $\beta$ -coboundary expander.*

The purpose of this subsection is to prove that the spherical buildings are also good skeleton expanders (good means that the normalized non-trivial eigenvalues approaches zero when the thickness degree is very large).

**Theorem 5.18** (Spherical buildings are spectral expander). *Let  $X$  be a  $d$ -dimensional  $q$ -thick spherical building which posses a strongly transitively action. Then  $X$  is an  $\frac{C(d)}{\sqrt{q}}$ -skeleton expander, where  $C(d) := \max\{2^d \cdot (d+1)!, 192 \cdot 11!\}$ .*

**Remark 5.19.** *Another way of stating Theorem 5.18 is to say that each spherical building which posses a strongly transitively action, satisfy a form of Szemerédi's regularity Lemma, i.e. their exists a partition of the vertices such that the sets of edges between every two different parts are pseudo-random.*

*However, note that while spherical buildings are not sparse (i.e. bounded degree), they are also not dense (i.e. degree is approximately the number of vertices), they are somewhere in between these two notions.*

The strategy for proving Theorem 5.18 is as follows: First we define a property for graph (symmetric-convex) and show that such graph have good bound on their eigenvalues. Second we show that the type-induced graphs of a spherical building which posses a strongly transitively action satisfy this property.

**Definition 5.20** (Symmetric convex graph). *Let  $X = (V_1 \sqcup V_2, E)$  be a bipartite graph and let  $c \in \mathbb{N}$ . Say that  $X$  is a  $c$ -symmetric-convex graph, if for any  $v \in V_1$ , then:*

1. *The number of  $G_v$ -orbits in  $X$  is at most  $c$ .*
2. *There is a unique  $G_v$ -orbit of vertices in  $V_1$  of maximal distance from  $v$ , and a unique  $_v$ -orbit of vertices in  $V_2$  of maximal distance from  $v$ .*
3. *For any vertex  $u$ , which is not of maximal distance from  $v$ , the number of neighbors of  $w \sim u$ , such that  $\text{dist}(v, w) < \text{dist}(v, u)$ , is at most  $c$ .*

**Theorem 5.21** (Symmetric-convex imply spectral expander). *Let  $X$  be bipartite  $(k, k')$ -biregular  $c$ -symmetric-convex connected graph, then the normalized secon largest eigenvalue of  $X$  is bounded by,*

$$\lambda(X) \leq c \cdot \max\left\{\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k'}}\right\} \quad (5.1)$$

*Proof.* Let  $A = A_X \in \text{End}(\mathbb{C}^{V(X)})$  be the adjacency operator of  $X$ , and let  $\text{Spec}(A) = \{\lambda_n, \dots, \lambda_2, \lambda_1\}$  be its set of eigenvalues. Let us recall some basic facts (see [EGL, § 3]): since  $X$  is an undirected graph the operator  $A$  is self-adjoint hence  $\text{Spec}(A) \subset \mathbb{R}$ , the graph is bipartite so  $\lambda_{n-i+1} = \lambda_i$  for any  $i$ , and finally the graph is biregular so  $\lambda_1 = \sqrt{k \cdot k'}$ .

Note that an eigenvectors  $f_n, f_1 \in \mathbb{C}^{V(X)}$  of the eigenvalues  $\lambda_n, \lambda_1$  respectively, are non-zero on every vertex, and if  $f_2 \in \mathbb{C}^{V(X)}$  is an eigenvector of  $\lambda_2$ , one can pick some  $v \in V_1$  such that  $f_2(v) \neq 0$ .

For the above vertex  $v$ , let  $K = \text{stab}_G(v)$  be its stabilizer in  $G$ , and define the following directed-multi-graph  $\bar{X} = X/K$  as follows: The vertices of  $\bar{X}$  are the  $K$ -orbits ( $[u] = \{k(u) | k \in K\}$  for some  $u \in V(X)$ ) of the vertices of  $X$ , and the number of edges in  $\bar{X}$  from  $[u]$  to  $[w]$ , is equal to the number of edges in  $X$  between the vertex  $u$  and the set of vertices  $\{k(w) | k \in K\}$  (note that this is independent of the choice of  $u$ ). Finally, let  $\bar{A} \in \text{End}(\mathbb{C}^{V(\bar{X})})$  be the adjacency operator of  $\bar{X}$ , and let  $\text{Spec}(\bar{A}) \subset \mathbb{C}$  be its set of eigenvalues.

Note that since  $\bar{X}$  is directed, a priori there is no reason for the eigenvalues of  $\bar{A}$  to be real, however: If  $\lambda$  is an eigenvalue of  $\bar{A}$ , and  $f \in \mathbb{C}^{V(\bar{X})}$  is his eigenvector, then defining  $f' \in \mathbb{C}^{V(X)}$  by  $f'(u) = f([u])$ , we get that for any  $v \in V(X)$ ,

$$A f'(u) = \sum_{w \in V(X)} A_{u,w} f'(w) = \sum_{[w] \in V(X/K)} \bar{A}_{[u],[w]} f([w]) = \bar{A} f([u]) = \lambda \cdot f([u]) = \lambda \cdot f'(u) \quad (5.2)$$

hence,  $f'$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ . In particular,

$$\text{Spec}(\bar{A}) \subset \text{Spec}(A) \subset \mathbb{R} \quad (5.3)$$

On the other hand, if  $\lambda \in \text{Spec}(A)$ , with an eigenvector  $f \in \mathbb{C}^{V(X)}$  such that  $f(v) \neq 0$ , then defining  $\bar{f} \in \mathbb{C}^{V(\bar{X})}$  by  $\bar{f}([u]) = \frac{1}{|K|} \sum_{k \in K} f(k(u))$ , we get that  $\bar{f} \neq 0$  and for any  $[u] \in V(\bar{X})$ ,

$$\begin{aligned} \bar{A} \bar{f}([u]) &= \sum_{[w] \in V(\bar{X})} \bar{A}_{[u],[w]} \bar{f}([w]) = \sum_{[w] \in V(\bar{X})} \bar{A}_{[u],[w]} \frac{1}{|K|} \sum_{k \in K} A f(k(w)) \\ &= \frac{1}{|K|} \sum_{k \in K} \sum_{w \in V(X)} A_{u,w} f(k(w)) = \frac{1}{|K|} \sum_{k \in K} A f(k(u)) = \frac{1}{|K|} \sum_{k \in K} \lambda \cdot f(k(u)) = \lambda \cdot \bar{f}([u]) \end{aligned} \quad (5.4)$$

hence  $\bar{f}$  is an eigenvector of  $\bar{A}$  with eigenvalue  $\lambda$ . In particular,

$$\lambda_n, \lambda_2, \lambda_1 \in \text{Spec}(\bar{A}) \subset \mathbb{R} \quad (5.5)$$

and by applying the trace formula for  $\bar{A}^2$ , we get

$$\lambda_2^2 + 2 \cdot k \cdot k' = \lambda_n^2 + \lambda_2^2 + \lambda_1^2 \leq \sum_{\lambda \in \text{Spec}(\bar{A})} \lambda^2 = \text{tr}(\bar{A}^2) \quad (5.6)$$

Finally, let us use the properties of the symmetric-convex graph: By property 1 the number of vertices in  $\bar{X}$  is at most  $c$ . By property 2 in each of the two parts of  $\bar{X}$ , there is a unique vertex,  $[v_i]$ ,  $i = 1, 2$ , of maximal distance from  $[v]$ , and since  $X$  is  $(k, k')$ -biregular, so does  $\bar{X}$ , and hence  $[v_1], [v_2]$  has at most  $k \cdot k'$  directed 2-paths starting and ending with them. By property 3 for any other vertex  $[u]$  in  $\bar{X}$ ,  $[u] \neq [v_1], [v_2]$ , the number of directed 2-paths starting and ending with  $[u]$  is at most  $c \cdot \max\{k, k'\}$ , since such a 2-path corresponds to a following 2-path in  $X$ ,  $u \sim w \sim u' = k(u)$  for some vertex  $w$  and  $k \in K$ , so either  $\text{dist}(v, u) < \text{dist}(v, w)$

in which case there are at most  $c$  such possible  $u' = k(u)$  (note  $\text{dist}(v, u) = \text{dist}(v, k(u))$ ), or  $\text{dist}(v, u) > \text{dist}(v, w)$  in which case there are at most  $c$  such possible  $w$ . All in all we get that

$$\text{tr}(\bar{A}^2) = \sum_{[u] \in \bar{X}} \#\{\text{directed 2-paths starting and ending with } [u]\} \leq 2 \cdot k \cdot k' + (c-2) \cdot c \cdot \max\{k, k'\} \quad (5.7)$$

Combining this with (5.2), we get

$$\lambda_2^2 \leq \text{tr}(\bar{A}^2) - 2 \cdot k \cdot k' \leq c^2 \cdot \max\{k, k'\} \quad (5.8)$$

and noting that  $\lambda(X) = \frac{\lambda_2}{\lambda_1} = \frac{\lambda_2}{\sqrt{k \cdot k'}}$ , finishes the proof.  $\square$

In the following Proposition we show that the type-induced bipartite graphs of the spherical buildings are symmetric-convex, with a constant  $c$  that depends only on the dimension (and not on the thickness). of the building.

**Proposition 5.22** (Building are symmetric-convex). *Let  $B$  be a  $d$ -dimensional  $q$ -thick spherical building which posses a strongly transitive action. Let  $i, j \in [d]$  be two types in of the building, and let  $B_{(i,j)}$  be the  $(i, j)$ -type induced bipartite biregular graph of  $B$ .*

*Then  $B_{(i,j)}$  is an  $C(d)$ -symmetric-convex graph, and both his regularity degrees are at least  $q+1$  (where  $C(d) := \max\{2^d \cdot (d+1)!, 192 \cdot 11!\}$ ).*

*Proof.* Let  $v$  be a vertex in the building,  $G_v = \text{stab}(v)$ , and let  $A$  be some fixed apartment that contains  $v$ .

First, let us prove the claim on the regularity degrees: Assume  $v$  is of type  $i$  (in the  $(d+1)$ -type function of Lemma 5.7), so it must be contained in some panel of cotype  $j$  (i.e. the panel has a vertex of each of  $d$  types except for the type  $j$ ),  $\sigma$ , and by the  $q$ -thickness  $\sigma$  is contained in  $q+1$  chambers,  $C_0, \dots, C_q$ , each of which contains a unique  $j$ -type vertex,  $u_0, \dots, u_q$ , which is a neighbour of  $v$  in the graph  $B_{(i,j)}$  (and of course the same reasoning apply when replacing  $i$  and  $j$ ).

1) By Lemma 5.15 we get that the number of  $G_v$ -orbits is at most the size of an apartment in the building, and by Lemma 5.5 we get that this number is bounded by  $C(d)$ .

2) Let  $C$  be some chamber in  $A$  which contains  $v$ . By Lemma 5.10 there is a unique chamber  $C_A^{op} \in A$  of maximal gallery distance from  $C$  in  $A$ , and let  $e = \{v_1, v_2\} \subset C^{op}$  be (the unique) edge of type  $\{i, j\}$  inside it. Now, since gallery distance is coarser then graph distance (any gallery path contains in it a graph path), then  $v_1, v_2$  are the two farthest vertices from  $v$  of type  $i, j$  respectively, inside the apartment  $A$ , w.r.t. the graph  $B_{(i,j)}$ . On the other hand, since  $G_v$  is a collection of automorphisms, hence preserves distances, and by Lemma 5.10 their is a unique chamber  $C_{A'}^{op}$  of maximal gallery distance from  $C$  in each apartment  $A'$ , we get that for any two apartments  $A', A''$ , there is some  $k \in G_v$  such that  $k(C_{A'}^{op}) = C_{A''}^{op}$ . So,  $[v_1]$  is the unique  $G_v$ -orbit in  $V_i$  of maximal distance vertices from  $v$ , and similar for  $[v_2]$  in  $V_j$ .

3) Since  $u$  is not of maximal distance from  $v$ , there is some neighbour  $w^*$  of  $u$  such that  $\text{dist}(v, u) < \text{dist}(v, w^*)$ . Now by the axiom of the building let  $A$  be an apartment that contains both  $v$  and the edge  $(u, w^*)$ . Let  $w_1, \dots, w_n$  be all the vertices in  $B_{(i,j)}$ , which are neighbours of  $u$  and satisfy  $\text{dist}(v, w_i) < \text{dist}(v, u)$  for any  $i = 1, \dots, n$ . Then, any minimal path from  $v$  to  $w^*$  which passes through  $u$ , most also pass through some  $w_i$ . Hence, since gallery distance is coarser then graph distance, any minimal gallery in the building from a maximal face containing  $\{v\}$ , to a maximal face containing  $\{u, w^*\}$ , most also pass through a maximal face containing  $\{u, w_i\}$  for some  $i$ . Now, if  $A$  is an apartment containing  $v$  and  $\{u, w^*\}$ , then by Lemma 5.9 all these minimal galleries lies in  $A$ , in particular all  $w_1, \dots, w_n$  lies in  $A$ , i.e.  $n \leq |A| \leq C(d)$  (where the last inequality is Lemma 5.5).  $\square$

Finally, Theorem 5.18 follows from Proposition 5.22 and Theorem 5.21.

*Proof of Theorem 5.18.* By Proposition 5.22 each type induced bipartite graph,  $B_{(i,j)}$ , of the building is  $C(d)$ -symmetric-convex. By Theorem 5.21 the building has normalize largest non-trivial eigenvalue at most  $\frac{C(d)}{\sqrt{q}}$ . Finally applying Proposition 3.4 we get the building is an  $\frac{C(d)}{\sqrt{q}}$ -skeleton expander.  $\square$

## 6 Ramanujan complexes

Ramanujan complexes were first defined in [LSV1], and were explicitly constructed in [LSV2]. They are finite quotients of affine buildings of type  $\tilde{A}$  (in particular, they posses a strong transitive action), which exhibit excellent spectral properties. For more on Ramanujan complexes, we refer the readers for the survey [L2].

First let us show that Ramanujan complexes satisfy the criterion of Theorem 4.1.

**Lemma 6.1** (Ramanujan skeleton Lemma). *Let  $X$  be a  $d$ -dimensional  $q$ -thick Ramanujan complex. Then, in the notation of § 3,  $\lambda(X) \leq d^d \cdot q^{-d/2}$ .*

*Proof.* Using the notation of [L2, § 2.1], then the adjacency operator of the type induced graph  $X_{i,j}$  is the Hecke operator  $A_{j-i}$ , hence by [L2, Remark 2.1.5] the above bound on the eigenvalue follows.  $\square$

**Lemma 6.2** (Ramanujan link Lemma). *Let  $X$  be a  $q$ -thick Ramanujan complex. Then each (proper) link of it is a  $q$ -thick spherical building which posses a strongly transitive action.*

*Proof.* Since a Ramanujan complex is a quotient of an affine building, then it is enough to prove the claim for  $q$ -thick affine building. Now the claim follows from Lemma 5.8 and 5.16, along with the observation that an affine building is locally finite so its link must be finite hence a spherical building.  $\square$

Combining these Lemmas with the results from the previous section yields.

**Theorem 6.3.** *Let  $X$  be a  $d$ -dimensional  $q$ -thick Ramanujan complex. Then  $X$  is a  $\beta$ -link coboundary expander and a  $(C \cdot q^{-1/2})$ -skeleton expander (where  $\beta = \beta(d)$  and  $C = C(d)$  are as in Theorems 5.17 and 5.18).*

*Proof.* By Lemma 6.2 and Theorem 5.17, we get that  $X$  is a link coboundary expander. By Lemmas 6.1, 6.2, Theorem 5.18 and Proposition 3.4, we get that  $X$  is a skeleton expander.  $\square$

**Remark 6.4.** *The above Theorem 6.3, should hold for any finite quotient of an affine building, not just for Ramanujan complexes. To prove this generalization, one needs only to prove Lemma 6.1 for the 1-skeleton of a quotient of any affine building. To do this one should use explicit property (T) (such as [Oh]), instead of the Ramanujan conjecture.*

Consequentially, applying Theorem 4.1 on the Ramanujan complexes, gives:

**Corollary 6.5.** *For any  $d \in \mathbb{N}$  there exists constants  $\epsilon = \epsilon(d) > 0$ ,  $\mu = \mu(d) > 0$  and  $q_0 = q_0(d) > 0$ , such that: Let  $\mathcal{B}$  be a  $(d+1)$ -dimensional  $q$ -thick affine building of type  $\tilde{A}$ , where  $q \geq q_0$ . Let  $\{X_n\}_n$  be an infinite family of Ramanujan complexes which are quotients of the building  $\mathcal{B}$ , and let  $Y_n$  be the  $d$ -dimensional skeleton of  $X_n$ . Then  $\{Y_n\}_n$  is an infinite family of  $d$ -dimensional bounded degree  $(\epsilon, \mu)$ -cosystolic expanders.*

Recall the following Gromov's criterion for topological overlapping property (see the introduction).

**Theorem 6.6** (Gromov's TOP criterion - see [DKW]). *Let  $X$  be a  $d$ -complex which is  $Q$ -bounded degree (i.e.  $|X_v| \leq Q$  for any  $v \in X(0)$ ) and  $(\epsilon, \mu)$ -cosystolic expander. Then, there exist  $c = c(d, \mu, \epsilon, Q) > 0$ , such that  $X$  has the  $c$ -topological overlapping property (see introduction). In particular, a family of bounded degree cosystolic expanders have the topological overlapping property.*

So, combining Gromov's TOP criterion with Corollary 6.5, yields.

**Corollary 6.7.** *Let  $d \in \mathbb{N}$  and let  $q \gg d$  be a prime power. Let  $B$  be a  $(d+1)$ -dimensional  $q$ -thick affine building of type  $\tilde{A}$ . Let  $\{X_n\}_n$  be an infinite family of Ramanujan complexes which are a quotients of the building  $B$ , and let  $Y_n$  be the  $d$ -dimensional skeleton of  $X_n$ . Then  $\{Y_n\}_n$  is an infinite family of bounded degree  $d$ -complexes with the topological overlapping property.*

**Remark 6.8.** *Using the explicit construction of Ramanujan complexes done in [LSV2], leads to an explicit construction of expanders in every dimension.*

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